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A GENERAL SOLUTION FOR THE RECTANGULAR AIRFOIL IN SUPERSONIC FLOW*

BY

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Summary. The potential on a rectangular airfoil due to an arbitrarily prescribed motion at its surface is obtained by an operational solution of the linearized equations and subsequent comparison with the known solution in steady flow. It is shown that the result can be extended to more general planforms with the aid of the Lorentz transformation. Other methods of solution are noted.

1. Introduction. The problem of unsteady motion of a rectangular airfoil in supersonic flow has been treated in closed form by Goodman [1, 2], Miles [3, 4], Rott [5], Stewartson [6], and Stewart and Li [7, 8]. The results of refs. 1-6 are in mutual agreement, but those of Stewart and Li are believed to be incorrect due to their (in our opinion†) erroneous conclusion that Evvard's "equivalent area" concept is applicable to non-stationary flow, the derivation [9] of which has been criticized elsewhere [10].

The reason for the addition of the present paper to this already voluminous literature is to present a solution that is valid for an arbitrarily prescribed motion of the airfoil. In principle, the solution of ref. 3 is sufficiently general in virtue of Fourier's theorem (with reference to the time dependence) and the possibility of expanding velocity distributions of practical interest in powers of y^n, but the following solution has the advantage of exhibiting the dependence of the potential on the prescribed velocity in a more explicit form. Moreover, the end result affords an immediate extension to planforms with oblique edges. Finally, the result permits a precise extension of the equivalent area concept to non-stationary motion of a rectangular wing, albeit this interpretation does not permit further generalization to other planforms.

2. Formulation of the problem. We consider a quarter infinite airfoil whose projection on the plane $z = 0$ is bounded by $X = -Ut/l$ and $y > o$ in the *fixed*, right handed, dimensionless (with l as the characteristic length), Cartesian coordinates (X, y, z), U being the flight velocity and t the time. On the basis of the usual assumptions, the linearized equation for the velocity potential is the wave equation

$$\phi_{XX} + \phi_{yy} + \phi_{zz} = \phi_{TT} \tag{2.1}$$

$$T = at/l \tag{2.2}$$

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**This work was carried out while at Auckland University College, New Zealand under the Fulbright program.

†We have discussed this with Prof. Stewart, who also is now of this belief.

where a is the sonic velocity and T a dimensionless time. In the end results we shall exhibit ϕ as a function of the *moving* coordinates (x, y, z), related to the fixed coordinates (X, y, z) by the Gallilean transformation

$$\begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} 1 & M \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ lT/a \end{pmatrix} \tag{2.3}$$

but in carrying out the analysis it is expedient to introduce the modified Lorentz transformation

$$\begin{pmatrix} \xi \\ \tau \end{pmatrix} = \beta^{-1} \begin{pmatrix} 1 & M \\ M & 1 \end{pmatrix} \begin{pmatrix} X \\ T \end{pmatrix}, \qquad \beta = (M^2 - 1)^{1/2} \tag{2.4}$$

$$\psi(\xi, y, z, \tau) = \phi(x, y, z, t) \tag{2.5}$$

Transforming the original wave equation (1), we obtain the new wave equation (with the positions of the axial and time variables reversed)

$$\psi_{\tau\tau} + \psi_{yy} + \psi_{zz} = \psi_{\xi\xi} \tag{2.6}$$

which is conveniently regarded as the canonical form for non-stationary flow at supersonic speeds. [If in (4) we take $\beta = (1 - M^2)^{1/2}$, $M < 1$, we have the classical Lorentz transformation, and (1) remains invariant with respect to the position of the space and time coordinates. The application to subsonic airfoil theory has been discussed by Kussner [11].]

The boundary condition to be invoked at the airfoil is

$$\phi_z = -w(x, y, t); \qquad x > 0, y > 0, z = 0 \tag{2.7}$$

where w is the prescribed velocity, *positive down*. In the transformed coordinates, we write

$$\psi_z = -v(\xi, y, \tau); \qquad \xi > 0, y > 0, z = 0 \tag{2.8}$$

$$v(\xi, y, \tau) = w(x, y, t) = w[\beta\xi, y, (l/\beta a)(M\xi - \tau)] \tag{2.9}$$

In consequence of the symmetry of this boundary condition with respect to $z = 0$, the potential itself is anti-symmetric thereto, and we may restrict our consideration to the half space $z \geq 0$, provided that we pose the additional boundary condition

$$\psi = 0; \qquad \xi > 0, y < 0, z = 0 \tag{2.10}$$

In addition to the local boundary conditions (8) and (10), it is necessary to impose the requirement that ψ behave asymptotically as a disturbance originating at the wing and vanishing identically forward of the zone of action of the wing, toward which end it suffices to write

$$\psi = 0; \qquad \xi < 0 \tag{2.11}$$

together with the Sommerfeld finiteness and radiation conditions in the original coordinates (X, y, z, T), although the latter may not be necessary in all cases.

To complete the formulation, we note that the perturbation pressure at the upper surface of the wing is given by

$$p = -\rho_0 l(\phi_t)_X = -\rho_0 a\beta^{-1}(M\psi_\xi + \psi_\tau); \qquad \xi > 0,\, y > 0,\, z = 0+ \tag{2.12}$$

We now pose the following problem: given w, find ϕ on $z = 0+$ or, equivalently, find a solution to (6) satisfying the boundary conditions (8), (10) and (11), together with the appropriate conditions at infinity.

3. Reduction to steady flow problem. Let Ψ be the transform obtained by posing the time dependence exp $(i\kappa\tau)$ and taking a Laplace transform with respect to ξ, *viz.*

$$\exp(i\kappa\tau)\Psi(s, y, z, \kappa) = \mathfrak{L}_\xi\{\psi\} = \int_0^\infty e^{-s\xi}\psi(\xi, y, z, \tau)\, d\xi \tag{3.1}$$

with a similar representative, V, for v. Transforming the boundary value problem of the preceding section, we have

$$\Psi_{yy} + \Psi_{zz} - \lambda^2\Psi = 0, \qquad \lambda^2 = s^2 + \kappa^2 \tag{3.2}$$

$$\Psi_z = -V; \qquad y > 0, z = 0 \tag{3.3}$$

$$\Psi = 0; \qquad y < 0, z = 0 \tag{3.4}$$

We now consider the "quasi-steady" problem obtained by setting $\lambda = s$ ($\kappa = 0$) in (2) [but $\kappa \neq 0$ in (3)], which is tantamount to neglecting $\psi_{\tau\tau}$ in (2.6), thereby reducing the boundary value problem to one in steady flow. The solution to this reduced problem, which is obtained most directly by Evvard's method [12], is designated by ψ_0 and given at the wing (all subsequent potentials also are specified at $z = 0+$) by

$$\psi_0(\xi, y, 0+, \tau) = \frac{1}{\pi}\int_0^\xi d\mu \int_{|(\xi-\mu)-y|}^{(\xi-\mu)+y} [(\xi - \mu)^2 - (y - \eta)^2]^{-1/2} v(\mu, \eta, \tau)\, d\eta \tag{3.5}$$

the integration being carried out over the trapezoid bounded by the Mach lines passing through (ξ, y), the reflection of one of these $[\eta = y - (\xi - \mu)]$ in the side edge, and the leading edge.

Taking the Laplace transform of ψ_0, we have, by a slight extension of the Faltung heorem,

$$\Psi_0 = \mathfrak{L}_\xi\{\psi_0\} = \mathfrak{L}_\xi\mathfrak{L}_\mu\left\{\frac{1}{\pi}\int_{|\xi-y|}^{\xi+y} [\xi^2 - (y - \eta)^2]^{-1/2} v(\mu, \eta, \tau)\, d\eta\right\} \tag{3.6}$$

Carrying out the indicated operation on v and substituting λ for s in the operation with respect to ξ, we have

$$\Psi(s, y, 0+, \kappa) = \int_0^\infty \exp[-(s^2 + \kappa^2)^{1/2}\xi]\left\{\frac{1}{\pi}\int_{|\xi-y|}^{\xi+y} [\xi^2 - (y - \eta)^2]^{-1/2} V(s, \eta, \kappa)\, d\eta\right\} d\xi \tag{3.7}$$

where the phase of $(s^2 + \kappa^2)^{1/2}$ is chosen to ensure a positive real part.

4. Solution for harmonic time dependence. To effect the inverse Laplace transform of (3.7) we introduce the theorem [*cf.* p. 123 of ref. 13, a source hereafter denoted by MO]

$$\mathfrak{L}^{-1}\{F[(s^2 + \kappa^2)^{1/2}]\} = f(\xi) + \int_0^{\pi/2} f(\xi\cos\theta)\frac{\partial}{\partial\theta} J_0(\kappa\xi\sin\theta)\, d\theta \tag{4.1}$$

the application of which yields

$$\psi = \psi_0 + \frac{1}{\pi}\int_0^{\pi/2} d\theta \int_0^{\xi} \frac{\partial}{\partial\theta} J_0[\kappa(\xi - \mu)\sin\theta]\, d\mu$$
$$\cdot \int_{|(\xi-\mu)\cos\theta - y|}^{(\xi-\mu)\cos\theta + y} [(\xi - \mu)^2 \cos^2\theta - (y - \eta)^2]^{-1/2} v(\mu, \eta, \tau)\, d\eta \tag{4.2}$$

where the second term appears as a correction on the "quasi-steady" solution.

An alternative expression for ψ that separates out the "strip theory" contribution

$$\psi_s = \int_0^{\xi} J_0[\kappa(\xi - \mu)] v(\mu, y, \tau)\, d\mu \tag{4.3}$$

can be obtained from (2) after integrating by parts and noting that

$$\lim_{\theta\to\pi/2} \int_{|(\xi-\mu)\cos\theta - y|}^{(\xi-\mu)\cos\theta + y} [(\xi - \mu)^2 \cos^2\theta - (y - \eta)^2]^{-1/2} v(\mu, \eta, \tau)\, d\eta = \pi v(\xi, y, \tau) \tag{4.4}$$

whence

$$\psi = \psi_s - \frac{1}{\pi}\int_0^{\pi/2} d\theta \int_0^{\xi} J_0[\kappa(\xi - \mu)\sin\theta]\, d\mu$$
$$\cdot \frac{\partial}{\partial\theta}\int_{|(\xi-\mu)\cos\theta - y|}^{(\xi-\mu)\cos\theta + y} [(\xi - \mu)^2 \cos^2\theta - (y - \eta)^2]^{-1/2} v(\mu, \eta, \tau)\, d\eta \tag{4.5}$$

In the special case where v is assumed to be independent of y the η integration may be carried out explicitly, the result being

$$\psi = \psi_s - \frac{2}{\pi}\int_0^{\pi/2} d\theta \int_0^{\xi} J_0[\kappa(\xi - \mu)\sin\theta] v(\mu, \tau) \frac{\partial}{\partial\theta} \sin^{-1}[y/(\xi - \mu)\cos\theta]^{1/2}\, d\mu \tag{4.6}$$

where the arc sine is to be replaced by $\pi/2$ when its argument exceeds unity. Changing the variable of integration yields

$$\psi = \psi_s - \frac{2}{\pi}\int_0^{\xi} d\mu \int_0^{\cos^{-1}(y/\mu)^{1/2}} J_0[\kappa(\mu^2 - y^2 \sec^4\varphi)^{1/2}] v(\xi - \mu, \tau)\, d\varphi \tag{4.7}$$

The last result is essentially in the form given by Stewartson [6]. [We remark that it was the form of Stewartson's result that suggested the present approach, although the use of (8) in connection with supersonic airfoil theory is due to Magnaradze [14, 15] and has been applied previously to the rectangular wing by Galin [14], but without much progress, since he found it necessary to introduce Fourier series in ξ (*cf.* ref. 15).]

5. Arbitrary time dependence. Generalizing the result (4.2) we have, after Fourier transformation (MO119) and convolution,

$$\psi = \psi_0 + \frac{1}{\pi^2}\int_0^{\pi/2} d\theta \int_0^{\xi} d\mu \int_{|(\xi-\mu)\cos\theta - y|}^{(\xi-\mu)\cos\theta + y} [(\xi - \mu)^2 \cos^2\theta - (y - \eta)^2]^{-1/2}\, d\eta$$
$$\cdot \frac{\partial}{\partial\theta}\int_{\tau-(\xi-\mu)\sin\theta}^{\tau+(\xi-\mu)\sin\theta} [(\xi - \mu)^2 \sin^2\theta - (\tau - \zeta)^2]^{-1/2} v(\mu, \eta, \zeta)\, d\zeta \tag{5.1}$$

In connection with the ζ limits of integration, we remark that the maximum and minimum values of ζ permitted in v are ∞ and $\tau - M(\xi - \mu)$, corresponding to $-\infty$ and

present time after transformation back to the physical variables. Both of these limits being outside of the domain in which the inverse transform of $J_0\,[\kappa(\xi - \mu)\sin\theta]$, is non-vanishing, we choose the limits as shown, although v may vanish over some part of the latter region, as in transient problems.

A more convenient expression for ψ, obtained by introducing a trigonometric variable in place of ζ, is given by

$$\psi = \psi_0 + \frac{1}{\pi^2}\int_0^{\pi/2} d\theta \int_0^{\pi} d\chi \int_0^{\xi} d\mu \int_{|(\xi-\mu)\cos\theta - y|}^{(\xi-\mu)\cos\theta + y} [(\xi - \mu)^2\cos^2\theta - (y - \eta)^2]^{-1/2} \cdot \frac{\partial}{\partial\theta} v[\mu, \eta, \tau + (\xi - \mu)\sin\theta\cos\chi]\, d\eta \tag{5.2}$$

Numerous, alternative forms may be obtained by additional changes of variable and by integration by parts.

Finally, upon substituting the physical variables x, t, ϕ and w from (2.2, 2.3, 2.4, 2.5, 2.9) we obtain

$$\phi_0 = \frac{1}{\pi}\int_0^{x} d\mu \int_{|\beta^{-1}(x-\mu) - y|}^{\beta^{-1}(x-\mu)+y} [(x - \mu)^2 - \beta^2(y - \eta)^2]^{-1/2} \cdot w[\mu, \eta, t - (Ml/\beta^2 a)(x - \mu)]\, d\eta \tag{5.3}$$

$$\phi = \phi_0 + \frac{1}{\pi^2}\int_0^{\pi/2} d\theta \int_0^{\pi} d\chi \int_0^{x} d\mu \int_{|\beta^{-1}(x-\mu)\cos\theta - y|}^{\beta^{-1}(x-\mu)\cos\theta + y} [(x - \mu)^2\cos^2\theta - \beta^2(y - \eta)^2]^{-1/2} \cdot \frac{\partial}{\partial\theta} w[\mu, \eta, t - (l/\beta^2 a)(M + \sin\theta\cos\chi)(x - \mu)]\, d\eta \tag{5.4}$$

6. Oscillating wing. For the important special case of a wing undergoing the oscillating motion prescribed by

$$w(x, y, t) = Rl\{w(x, y)e^{i\omega t}\} \tag{6.1}$$

we have only to choose

$$\chi = kM/\beta, \qquad k = \omega l/U \tag{6.2}$$

in (4.3) and (4.5) and return to the original variables, whence

$$\phi_s = \beta^{-1} Rl\left\{\int_0^x \exp\,[i\omega t - i(kM^2/\beta^2)(x - \mu)] J_0[(kM/\beta^2)(x - \mu)] w(\mu, y)\, d\mu\right\} \tag{6.3}$$

$$\phi = \phi_s - \frac{1}{\pi} Rl\left\{\int_0^{\pi/2} d\theta \int_0^x \exp\,[i\omega t - i(kM^2/\beta^2)(x - \mu)] J_0[(kM/\beta^2)(x - \mu)\sin\theta]\, d\mu \cdot \frac{\partial}{\partial\theta}\int_{|\beta^{-1}(x-\mu)\cos\theta - y|}^{\beta^{-1}(x-\mu)\cos\theta + y} [(x - \mu)^2\cos^2\theta - \beta^2(y - \eta)^2]^{-1/2} w(\mu, \eta)\, d\eta\right\} \tag{6.4}$$

7. Extension of Evvard's "equivalent area" concept. It is of some interest to note that in (4.2) *et. seq.* the domain of the (μ, η) integration is bounded by the pseudo Mach lines $\eta = y \pm (\xi - \mu)\cos\theta$, together with the reflection of $\eta = y - (\xi - \mu)\cos\theta$,

namely $\eta = (\xi - \mu) \cos \theta - y$, in the side edge. This interpretation furnishes an extension of Evvard's "equivalent area" concept to unsteady flow, but the extension is not valid for oblique or curved side edges, since, in general, the equation for the reflected Mach line would be of the form $\eta = f(\xi, \xi - \mu) - y$, rather than simply $f(\xi - \mu) - y$, and the Faltung theorem [cf. (3.6)] would not be applicable.

8. Oblique edges. The extension of the foregoing results to wings having arbitrarily prescribed supersonic leading edges and streamwise side edges is trivial, since it is necessary only to circumscribe a fictitious rectangular wing and set $w = 0$ over those portions that are not included in the original planform. [However, it should be remarked that the Mach lines from the opposite corners may intersect on the wing only if it is rectangular (*cf.* ref. 3).] To extend the results to a straight, subsonic leading edge that is adjacent to a supersonic leading edge and does not interfere with any other subsonic edge, we may apply the Lorentz transformation

$$\begin{pmatrix} \xi' \\ y' \end{pmatrix} = (1 - m^2)^{-1/2} \begin{pmatrix} 1 & m \\ m & 1 \end{pmatrix} \begin{pmatrix} \xi \\ y \end{pmatrix}, \qquad m > 1 \tag{8.1}$$

under which (2.6) remains invariant. Unfortunately, it is then no longer possible to obtain results for the spanwise integrals of the potential that are comparable in simplicity to those for the rectangular wing (*cf.* refs. 3, 5, 6.).

The use of (1) in connection with supersonic wings in steady flow is well known (*cf.* refs. 16-18), but the application to unsteady flow problems seems not to have been noticed previously, perhaps due to the fact that the differential equation usually is written for $\phi(x, y, z, t)$, rather than $\psi(\xi, y, z, \tau)$.

9. Other methods. It is rather curious that of all the methods that have been applied to the problem of diffraction by a half plane, the classical counterpart of the supersonic rectangular wing, none [*e.g.*, Poincaré's original treatment, Sommerfeld's celebrated application of multivalued integrals, Lamb's use of parabolic coordinates, Magnus' solution of the integral equation, the Wiener-Hopf technique applied by Copson and Schwinger, *etc.*; *cf.* ref. 19] has proved as effective as the methods derived especially for the wing problem, notably Busemann's use of homogeneous solutions ("conical flows") and Evvard's method. (We remark that each of these methods had antecedents in the work of Bateman [20].) This is at least partially due to the different foci of interest in the two situations, *viz.* the potential on $z = 0$ in the wing problem and the solution at a distance (particularly near the boundary of the geometric shadow) in the diffraction problem, but it is of interest to note that the method of conical flows furnishes an elegant approach to the problem of pulse diffraction [21, 22, 23], while the application of Evvard's method to case of an arbitrary incident wave has received attention in a recent paper by Friedlander [24].

The foregoing remarks notwithstanding, Lamb's method has proved rather attractive for the special case of no spanwise variation of w, an application discussed recently by Rott [5]. This method is, in fact, applicable to more general spanwise distributions [25], but, in the form presented by Rott, it is much less direct than the present method, from which it differs fundamentally in prescribing the boundary data on the wing tip Mach cone, rather than the wing proper. However, if the potential is desired only at the wing the boundary data may be prescribed there. Thus, a solution to (3.2) in the polar coordinates (ρ, φ) that vanishes on $z = 0$, $y < 0$ $(\varphi = \pi)$, gives a null value

of the normal derivative on the wing ($\varphi = 0$) and vanishes properly at infinity is given by

$$\Psi_s = F(s, \kappa)\rho^{-1/2}e^{-\lambda\rho} \cos(\varphi/2). \tag{9.1}$$

This solution evidently is appropriate to the case where v is independent of y. Moreover, at large distances from the edge Ψ must reduce to the two dimensional solution, *viz.* [*cf.* ref. 6 or (3.2) and (3.3) with $\Psi_{yy} = 0$]

$$\lim_{y\to\infty} \Psi = \Psi_s = \lambda^{-1}v(s, \chi)e^{-\lambda|z|}. \tag{9.2}$$

Integrating (1) subject to the conditions that Ψ must vanish at $y = 0$ and satisfy (2) asymptotically, we have on $\varphi = 0$

$$\Psi = \lambda^{-1}v(s, \chi) \text{ erf } [(\lambda y)^{1/2}] \tag{9.3}$$

the inversion of which yields the result (4.7).

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REFLECTION OF WAVES FROM VARYING MEDIA*

BY

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Abstract. Formulae are found for the coefficient of reflection from varying media of a type encountered in physics. These are applied approximately for some general classes of media, and exactly for some specific cases. Many media which would normally be expected to be highly reflecting are shown to be completely transparent to certain waves at least and, in some cases, to a whole spectrum of waves. The results are considered both for electromagnetic (or other classical) waves and for mass waves.

I. Introduction and Summary. Many problems of wave propagation through varying media introduce the equation

$$\frac{d^2f}{dx^2} + g(x)f = 0 \tag{1}$$

for solution. This equation was studied some years ago by Lord Rayleigh [1] and others in connection with classical wave problems. The advent of quantum mechanics has led to a more thorough investigation of the solutions of (1), centered around the method developed by Brillouin [2], Wentzel [3] and Kramers [4]—the B.W.K. method. More recent work along these lines has been carried out by Kemble [5, 6], Langer [7, 8, 9], and Furry [10]. Exact solutions, in terms of hypergeometric functions, can be found if $g(x)$ is of the proper form; this method of attack was developed by Eckart [11] and Epstein [12]. In all these treatments, great attention has been paid to solutions obtained when $g(x)$ is positive for some ranges of x and negative for others. Such solutions are of particular interest in quantum mechanics, in some oblique incidence problems, and in problems of electromagnetic wave propagation through a conducting medium such as the ionosphere.

In the present paper we will deal with functions $g(x)$ which are everywhere real, finite, and continuous, have a constant positive value for large values of $|x|$, and lesser values in a region of variation about $x = 0$.

Solutions are obtained in terms of a new variable $r = r(x)$ which is itself a solution of a Riccati equation.† Exact formulae, also in terms of r, are developed for the reflection coefficient of a medium. The curve of $r(x)$ is traced by inspection of the determining differential equation. A surprisingly large amount of information about r can be found in this way, but the procedure is only qualitative.

By this method, then, we obtain an approximate evaluation of the exact solution, rather than an exact evaluation of an approximate solution as is found in the B.W.K. method. This difference of approach leads to a difference in applicability. The B.W.K. method is of prime importance in deriving numerical results in a particular given problem, although it has been used to infer some general results. The present method

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†See the paper by S. A. Schelkunoff [16] for comments on the occurrence of the Riccati equation in wave transmission problems and, indeed, for a number of interesting and pertinent remarks on the whole problem of propagation under varying conditions.

leads to the construction of problems to which it can give an exact solution, predicts some general results, and indicates some surprising (?) results which can be obtained if the medium is of the proper form. In particular, it is shown that many media which might be expected, by inference, to be highly reflecting can be completely transparent to one frequency (or energy) at least. This is particularly true of media in which $g(x)$ is negative in two separated regions; in this case a whole spectrum may be transmitted.

The formulae are developed in the symbolism commonly used in electromagnetic wave propagation, but the results are also considered in terms of mass waves.

II Electromagnetic Waves. In the electromagnetic wave problem, an equation of the form of (1) comes from eliminating time, t, from

$$\frac{\partial^2\Psi}{\partial x^2} - \frac{n^2}{c^2}\frac{\partial^2\Psi}{\partial t^2} = 0 \tag{2}$$

by setting

$$\Psi = f(x)\epsilon^{-ikct}. \tag{3}$$

One thus obtains

$$\frac{d^2f}{dx^2} + k^2n^2f = 0. \tag{4}$$

Here $n^2 = n^2(x)$ is the effective permittivity, c is the vacuum speed of light, and k is the (real positive) circular wave number in vacuum.

We shall here be concerned only with media for which $n^2 = n^2(x)$ is real, finite, continuous, and such that

$$\begin{aligned} n &= 1 \qquad \text{for} \qquad |x| > x_a > 0, \\ n^2 &\leq 1 \qquad \text{for} \qquad |x| < x_a\,. \end{aligned} \tag{5}$$

These conditions are imposed primarily for convenience of discussion, and most may be removed if proper modifications in the development are made.

Introduce $r = r(x)$ by defining

$$f = \exp\left[ik\int_{x_0}^{x} r(s)\,ds\right] \tag{6}$$

where x_0 is an arbitrary real constant. Substituting this into (4) we get

$$n^2 = r^2 - \frac{i}{k}\frac{dr}{dx} \tag{7}$$

as the equation determining $r(x)$. We shall allow only continuous solutions of (7) so that f and df/dx will be continuous. Three solutions of (7) will be of particular interest to us; they will be denoted r^+, r^-, and r', and are characterised by the following conditions:

$$r^+ = 1 \qquad \text{for} \qquad x \leq -x_a\,, \tag{8}$$

$$r^- = -1 \qquad \text{for} \qquad x \leq -x_a\,, \tag{9}$$

$$r' = 1 \qquad \text{for} \qquad x \geq x_a\,. \tag{10}$$

Writing

$$r^+ = r_1 + ir_2 \tag{11}$$

where r_1 and r_2 are real, equation (7) yields

$$n^2 = r_1^2 - r_2^2 + \frac{1}{k}\frac{dr_2}{dx}, \tag{12}$$

$$0 = 2r_1r_2 - \frac{1}{k}\frac{dr_1}{dx}, \tag{13}$$

since we are taking n^2 to be real. Equation (13) gives immediately

$$r_2 = \frac{1}{2k}\frac{d}{dx}\log|\,r_1\,| \tag{14}$$

by which we obtain

$$\int_{x_0}^{x} r^+(s)\,ds = \int_{x_0}^{x} r_1(s)\,ds + \frac{i}{2k}\log\left|\frac{r_1(x)}{r_1(x_0)}\right|. \tag{15}$$

For convenience, take $x_0 = -x_a$ so that, by (8) and (11)

$$r_1(x_0) = r_1(-x_a) = 1, \tag{16}$$

and hence we can write

$$f^+ = \frac{1}{\sqrt{|\,r_1(x)\,|}}\exp\left[ik\int_{-x_a}^{x} r_1(s)\,ds\right] \tag{17}$$

as a solution of (4). It will be shown in Part IV that $r_1(x)$ is always positive, hence the absolute value bars may be dropped.

The expression

$$F^+f^+\epsilon^{-ikct} = \frac{F^+}{\sqrt{r_1(x)}}\exp\left[ik\int_{-x_a}^{x} r_1(s)\,ds - ikct\right] \tag{18}$$

with F^+ an arbitrary complex constant, will then be a solution of (2). In the region to the left of the variation (i.e. for $x < -x_a$, by convention) this represents a wave of amplitude F^+ and speed c moving to the right. In the region to the right of the variation ($x > x_a$), it must represent, in general, two waves, one moving to the right and the other to the left, each with speed c.

It can be seen by inspecting (12) and (14) that if $r_1 + ir_2$ is a solution of (7), so too is $-r_1 + ir_2$; this is, in fact, simply r^-. Hence we have a second solution of (2) namely

$$F^-f^-\epsilon^{-ikct} \equiv \frac{F^-}{\sqrt{r_1(x)}}\exp\left[ik\int_{-x_a}^{x} r_1(s)\,ds - ikct\right]. \tag{19}$$

Remarks similar to those above apply to this solution also.

The two solutions obtained are independent, the Wronskian of the system f^+, f^- having the constant value $-2ik$, hence the general solution of (2) is

$$\Psi = (F^+ f^+ + F^- f^-)\epsilon^{-ikct} \tag{20}$$

with F^+ and F^- arbitrary complex constants.

Another solution of (2) is

$$F' f' \epsilon^{-ikct} \equiv \frac{F'}{\sqrt{r_1'(x)}} \exp\left[ik \int_{x_a}^{x} r_1'(s)\, ds - ikct\right]. \tag{21}$$

Here we have taken $x_0 = x_a$ and $r' = r_1' + ir_2'$, with r_1' and r_2' real. This solution represents, in the region to the right of the variation, a single wave moving to the right.

III. Formulae for Reflection Coefficients. In obtaining reflection coefficients, we consider solutions of (2) which give a single (transmitted) wave leaving the region on one side or the other. If this occurs on the left we must have $F^+ = 0$, and the complete solution is then given by (19). If the single wave moves off on the right, then the ratio F^-/F^+ must be such that the left-moving components on the right cancel out. In this case, however, we can equally well give the complete solution by (21). For equivalence,

$$F^+ f^+ + F^- f^- = F' f' \tag{22}$$

and, differentiating this with respect to x and dividing out ik,

$$r^+ F^+ f^+ + r^- F^- f^- = r' F' f'. \tag{23}$$

From these we obtain the complex amplitude reflection coefficient

$$\begin{aligned} R \equiv \frac{F^-}{F^+} &= -\frac{r^+ - r'}{r^- - r'} \frac{f^+}{f^-} \\ &= -\frac{(r_1 - r_1') + i(r_2 - r_2')}{(-r_1 - r_1') + i(r_2 - r_2')} \exp\left[2ik \int_{-x_a}^{x} r_1(s)\, ds\right] \end{aligned} \tag{24}$$

which has modulus

$$|R| = \left| \frac{(r_1 - r_1') + i(r_2 - r_2')}{(r_1 + r_1') + i(r_2 - r_2')} \right|. \tag{25}$$

Since R is a constant, the evaluation may be made at any x.

If the variation is symmetrical about $x = 0$—i.e.,if $n^2(x) = n^2(-x)$—then it may easily be shown that

$$r_1'(x) = r_1(-x); \qquad r_2'(x) = -r_2(-x). \tag{26}$$

In particular, at $x = 0$ we get $r_1' = r_1$ and $r_2' = -r_2$. Substituting this in (25), we get

$$|R_s| = [1 + r_1^2(0)/r_2^2(0)]^{-1/2} \tag{27}$$

for a symmetrical medium.

It may be noted that formulae (22) to (25) would be valid even if n^2 were complex in the region $|x| < x_a$

IV. Approximate Curves for r^+ (x). No general solution of the Riccati equation is known, so we are apparently no further ahead. However, the sort of solution to be expected can be estimated, to a certain extent, by inspection of the equations (12) to (14), and this task will be undertaken in the present part. The development is necessarily lengthy, and has only been outlined here. However qualitative it may be, it does lead to some general results and does provide a guide for further extension, as illustrated in succeeding parts. Moreover, since the required curves are shown to be smooth, at least over the regions of interest, it appears that electronic computers might be used to advantage in this sort of approach to the problem. (Contrast the actual wave solution which is, in the main, oscillatory.) The operator D will be used for d/dx throughout the discussion.

We shall be concerned here primarily with fairly slowly varying media, having simple curves of $n^2(x)$, in which n^2 is negative over some range, say $x_b < x < x_d$. A representation of such a medium is shown in Fig. 1. Media in which n^2 is slowly varying but never becomes negative will be classified as Type I, and will receive only scant treatment, near the end.

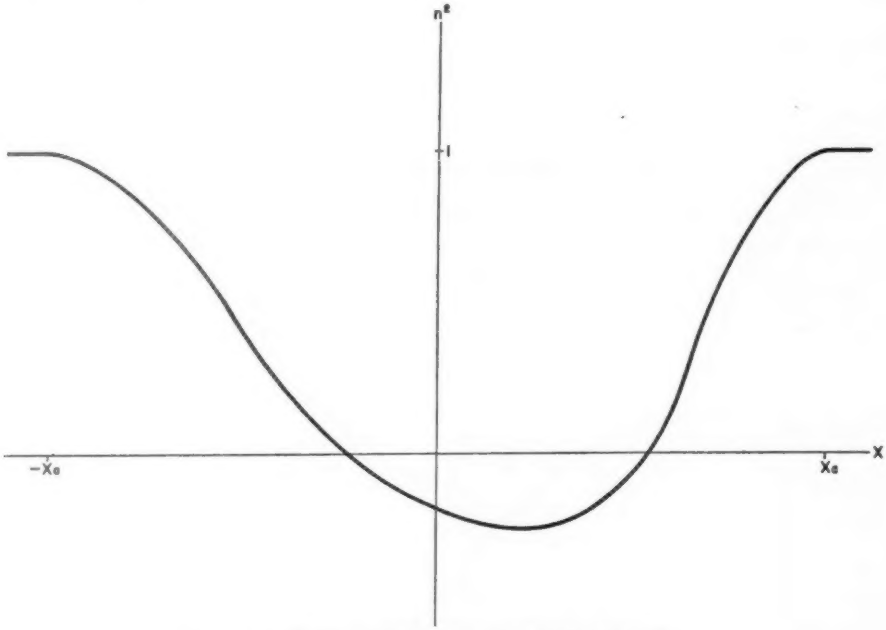

Fig. 1 TYPICAL MEDIUM TO BE CONSIDERED

We start from the region $x < -x_a$, where $r_1 = 1$ and $r_2 = 0$, and proceed to the right. Since r^+ is continuous, no change can occur in either r_1 or r_2 until one occurs in Dr_2. Since n^2 starts to decrease, we see from (12) that Dr_2 must become negative. Then D^2r_1 and Dr_1 become negative, and r_1^2 decreases from unity. So long as $3(Dr_1)^2 > 2r_1D^2r_1$, however, r_1^2 must stay greater than n^2, as may be seen on eliminating r_2 from (12) and (14). This gives, for the start of the r_1^2 and $r_2 \mid r_2 \mid$ curves the form shown in

Fig. 2. (We plot $r_2 \mid r_2 \mid$ so that the magnitude of r_2^2 and the signs of r_2 and Dr_2 may be shown.)

Before proceeding farther, we might first note two general properties of the r_1^2 curves, based on (12) and (14):

i) r_1 will never become zero (if x_a is finite) for this would require $r_2 \to -\infty$, and hence $n^2 \to -\infty$, a case which has been excluded from the present paper.

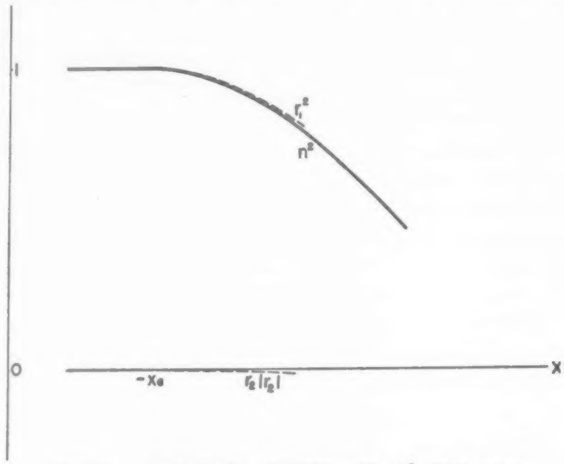

Fig. 2 STARTING CURVES OF r_1^2 AND $r_2|r_2|$

ii) if ever Dr_1^2 becomes zero after having been negative, so too will Dr_1, hence at that point $r_2 = 0$ and $Dr_2 > 0$, and so $n^2 > r_1^2$. Hence, so long as r_1^2 remains greater than n^2, it will not have a zero derivative.

For slowly varying media we may expect slowly varying r curves, in which case we will have $r_1^2 \approx n^2$ until very close to x_b, where n^2 vanishes. Such a result is, in fact,

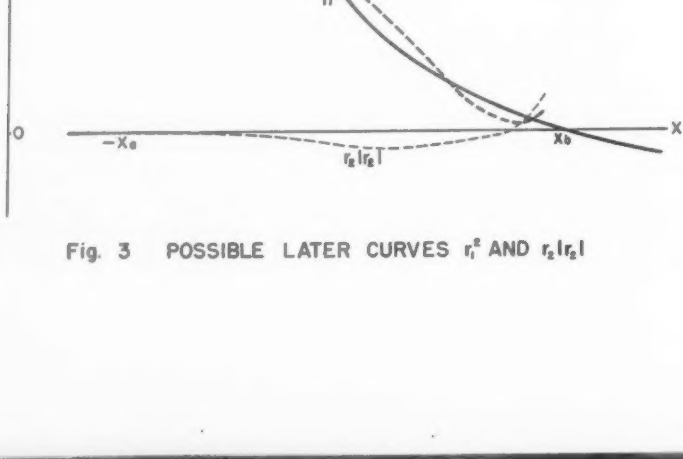

Fig. 3 POSSIBLE LATER CURVES r_1^2 AND $r_2|r_2|$

employed in the B.W.K. method where, in effect, $r_1 \approx n$ (for $x < x_b$) is derived from the first term of a series solution of (12) and (14), and then employed in (18). Except in the limit of vanishing variation per wave length ($k \to \infty$) we cannot know that this result must be obtained, but the reverse certainly is true: if r_1 and Dr_1 are very slowly varying, then n^2 is too, and $r_1^2 \approx n^2$. Later curves of r_1^2 and $r_2 \mid r_2 \mid$ may then be like those in Fig. 3 or, more likely if the medium is slowly varying, like those in Fig. 4 or 5.

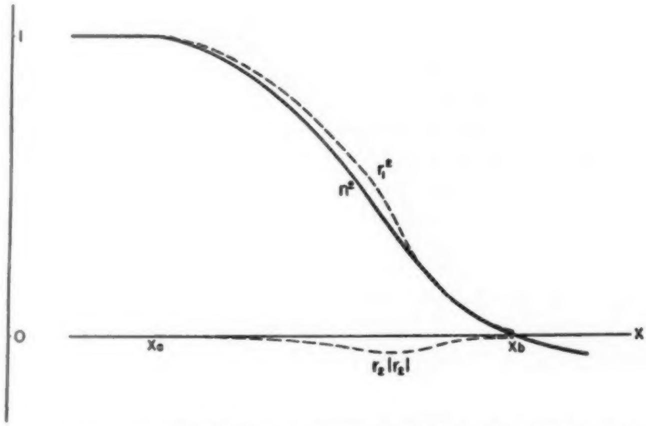

Fig. 4 POSSIBLE LATER CURVES OF r_1^2 AND $r_2|r_2|$

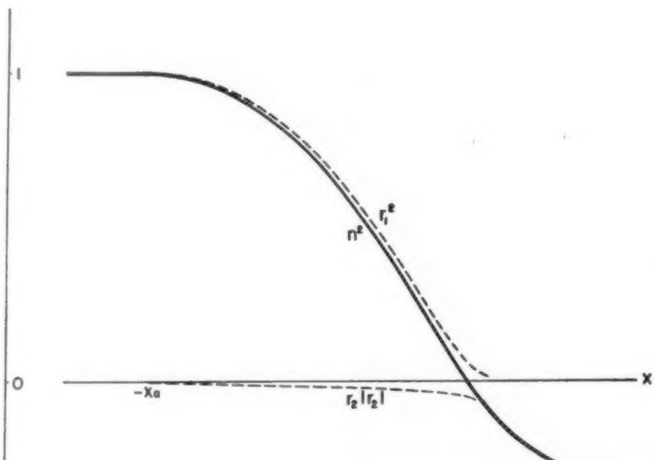

Fig. 5 POSSIBLE LATER CURVES OF r_1^2 AND $r_2|r_2|$

At any rate, when we come to x_b we have $n^2 = 0$, $r_1^2 > 0$. Media having $Dr_1 > 0$ at x_b will be classified as Type II while those having $Dr_1 < 0$ there will be classified as Type III. There does not appear to be any general way of continuing the curves past

x_b for media of Type II, which we shall therefore drop from our discussion for the present. Type III can, however, be treated generally, as follows:

As before, there will be no point in the region $x_b < x < x_d$ where $Dr_1 = 0 = r_2$, since the property (ii) above applies to Type III media there. This means, then, that $Dr_1 < 0$ and $r_2 < 0$ throughout this region. Moreover, if the medium is sufficiently slowly varying, r_1^2 will not differ from n^2 by very much in the region to the left of x_b, and hence will be very small at, and to the right of, x_b. In fact, it can be shown that

$$\int_{x_b}^{x_d} r_1 \, dx < \pi/2k \tag{28}$$

for any medium of Type III. (This is easily deduced from the fact that the graph of the real part of f must be curved away from the x-axis in regions of negative n^2.) From this we see that r_1 must become quite small between x_b and x_d if $k(x_d - x_b)$ is reasonably large, no matter how large r_1 is at x_b.

Proceeding to the right, we see that if $-r_2^2$ is less than n^2 by more than the (small) amount r_1^2, then Dr_1 must be positive and $-r_2^2$ must be increasing. So long as n^2 is not increasing, then, $-r_2^2$ cannot be much less than n^2 for long. Once $-r_2^2$ becomes greater than $n^2 - r_1^2$ it must remain there, at least until this latter starts to increase (near x_m, the point of minimum n^2). All that time Dr_2 will be negative, and so $-r_2^2$ will be decreasing. If the gap between $-r_2^2$ and $n^2 - r_1^2$ becomes large, so too will $-Dr_2$, thus forcing the gap to diminish (assuming fairly slow variation of n^2). Figs. 6 and 7 show the type of curves to be expected.

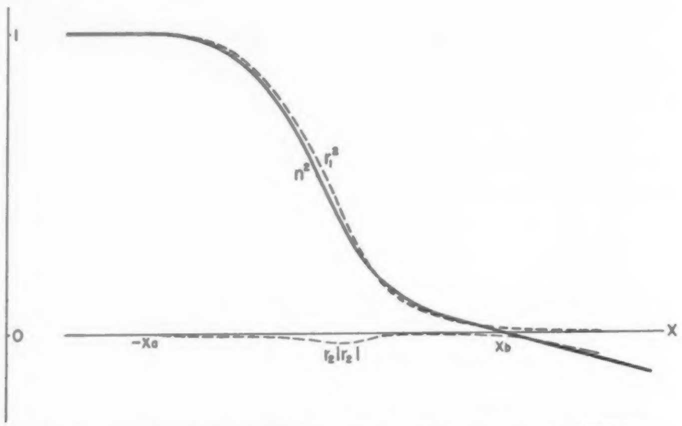

Fig. 6 POSSIBLE CURVES OF r_1^2 AND $r_2|r_2|$, SHOWING $-r_2^2$ BECOMING GREATER THAN n^2

All these considerations hold especially well when the variation is sufficiently slow. In the limiting case $k \to \infty$ we will have $r_1 \to n$, $r_2 \to 0$ for $x < x_b$, and $r_1 \to 0$, $r_2 \to -|n|$ for $x_b < x < x_d$. As we relax this condition of slow variation, the considerations above are modified. Inspection shows that this leads to curves such as those sketched in Figs. 8, 9, 10, and 11. They are, of course, only rough drawings meant to indicate the sort of curves to be expected; they are not mathematically accurate, and "slow variation" is not defined.

In most cases of slow variation, the curves can be extended as far to the right as

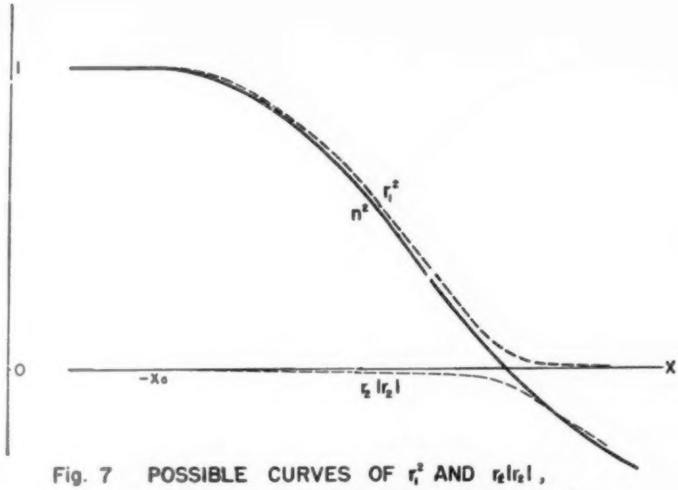

Fig. 7 POSSIBLE CURVES OF r_1^2 AND $r_2|r_2|$, SHOWING $-r_2^2$ BECOMING GREATER THAN n^2

x_d without too much error. After this point, however, it is impossible to continue by inspection, for r_1 and its derivatives become large. However, we do know that

$$\exp\left[ik\int_{-x_a}^{x} r^+(s)\,ds\right] = A\epsilon^{iknx} + B\epsilon^{-iknx} \tag{29}$$

for $x > x_a$. Here A and B are complex constants and, in our case, $n = 1$. From this can be derived

$$r^+(x) = n\,\frac{A\epsilon^{iknx} - B\epsilon^{-iknx}}{A\epsilon^{iknx} + B\epsilon^{-iknx}}. \tag{30}$$

In general (i.e. $B \neq 0$) this gives oscillatory curves for r_1 and r_2.

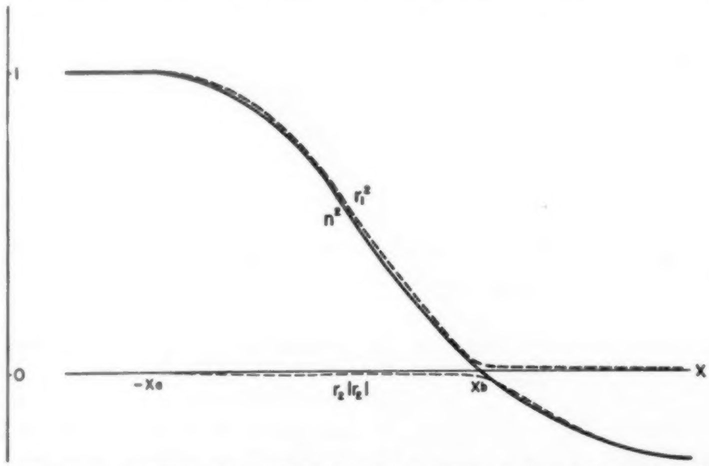

Fig. 8 r_1^2 AND $r_2|r_2|$ CURVES FOR A VERY SLOWLY VARYING MEDIUM OF TYPE III

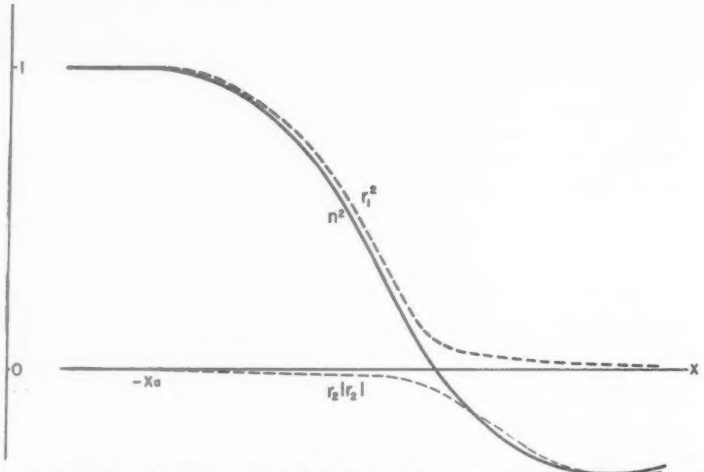

Fig. 9 r_1^2 AND $r_2|r_2|$ CURVES FOR A SLOWLY VARYING MEDIUM OF TYPE III

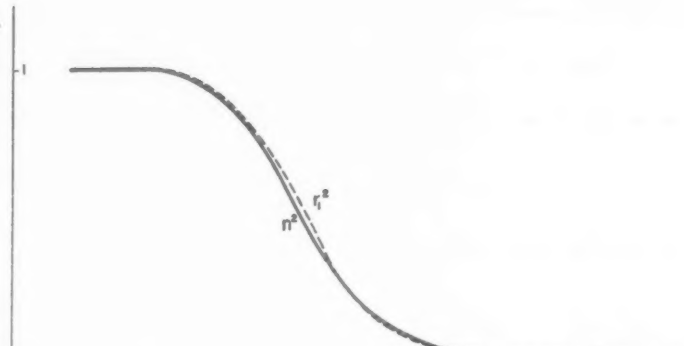

Fig. 10 r_1^2 AND $r_2|r_2|$ CURVES FOR A SLOWLY VARYING MEDIUM OF TYPE III

Actually, there is no need to describe the curves past x_m, since the formulae (25) and (27) can be used at x_m itself. We can trace the curves for $r_1'^2$ and $r_2' \mid r_2' \mid$, working to the left of x_a in a similar way, to the region of the minimum of n^2. This gives a complete set of curves for the medium, such as that shown for a typical case in Fig. 12. Other media will have curves differing in detail from these, but we may take it as a fairly general result that media having a single fairly wide region of negative n^2 will have r_1 and r_1' quite small, and $-r_2$ and r_2' approximately equal to $\mid n \mid$, at x_m.

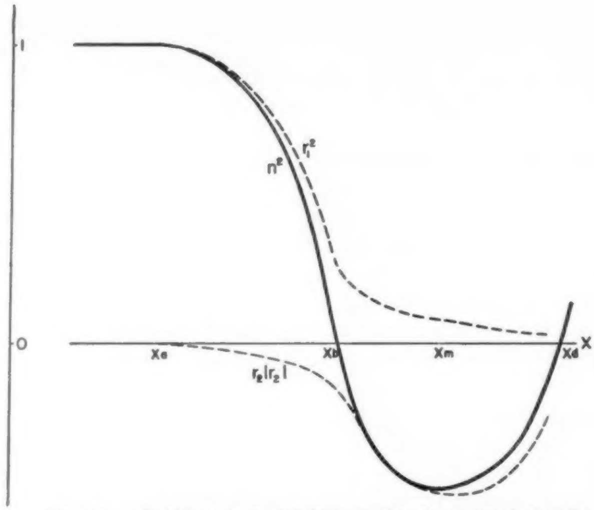

Fig. 11 r_1^2 AND $r_2|r_2|$ CURVES FOR A FAIRLY RAPIDLY VARYING MEDIUM OF TYPE III

This last statement will be true even for media of Type II, since it can be shown for them that

$$\int_{x_b}^{x_d} r_1 \, dx < \pi/k. \tag{31}$$

If, then, n^2 stays negative for long enough, r_1 must become very small. Along with

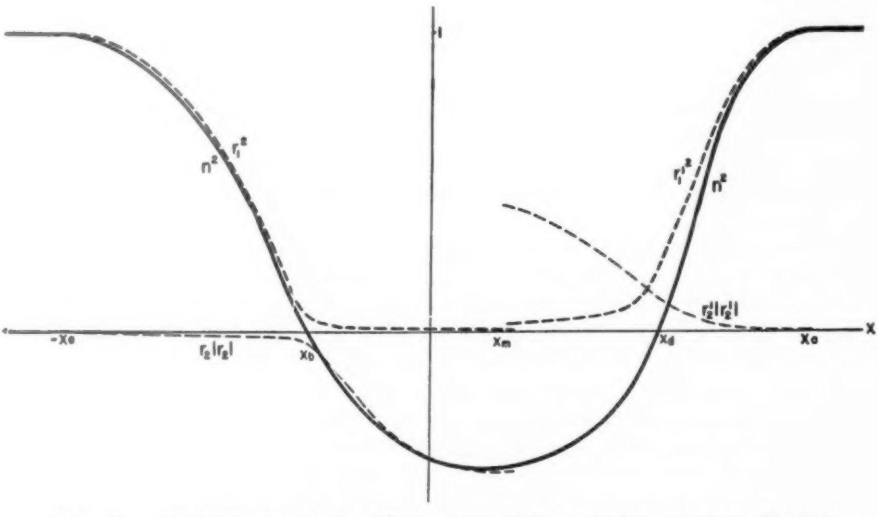

Fig. 12 CURVES OF r_1^2, $r_2|r_2|$, $r_1'^2$ AND $r_2'|r_2'|$ FOR A FAIRLY SLOWLY VARYING MEDIUM OF TYPE III ON BOTH SIDES

this, Dr_1 must become negative, and the results derived above for Type III media will become applicable.

In Type I media having n^2 slowly varying, we expect the approximation $r_1 \approx n$, $r_2 \approx 0$, to be fairly good throughout the entire range, at least if the minimum n^2 is not too close to zero.

V. Approximate Reflection Coefficients. i) *Type I media, minimum* $n^2 > 0$: So long as the approximations $r_1 \approx n \approx r_1'$, $r_2 \approx 0 \approx r_2'$ are good, (25) gives $| R | \approx 0$. If these approximations break down, as will happen in rapidly varying media, the resulting reflection coefficient may become appreciable.

ii) *Type I media, minimum* $n^2 = 0$: It can be seen that, under certain circumstances (e.g. $r_1 \approx -r_2$, $r_1' \approx r_2'$, $Dr_2 \approx 0 \approx Dr_2'$ at x_m) we will obtain $| R | \approx 1 \sqrt{2}$. This is a result frequently quoted for this case (e.g. Rydbeck [13]), but as can be seen, it is by no means general. An example will be given (Part IX, Fig. 20) in which it completely breaks down.

iii) *Media having a broad region of negative* n^2: We will have, at x_m, $-r_2 \gg r_1 > 0$, $r_2' \gg r_1' > 0$, and hence $| R | \approx 1$. We thus get the expected result, that such media are almost completely reflecting.

iv) *Symmetrical media*: On integrating (14) and introducing the result into (27) we obtain for symmetrical media

$$| R_s | = \left[1 + \frac{1}{r_2^2(0)} \exp \left\{ 4k \int_{-x_a}^{0} r_2 \, dx \right\} \right]^{-1/2}. \tag{32}$$

This gives a form which may be compared with the common (approximate) formula

$$| R | \approx \left[1 + \exp \left\{ -2k \int_{x_b}^{x_d} | n | \, dx \right\} \right]^{-1/2} \tag{33}$$

which becomes, for symmetrical media

$$| R_s | \approx \left[1 + \exp \left\{ -4k \int_{x_b}^{0} | n | \, dx \right\} \right]^{-1/2}.$$

This was originally developed by Gamow [14] by considering discontinuous media, and more recently by the "good path" method [6] [13]. If n^2 is sufficiently negative for a sufficient range of x, both methods give $| R_s | \approx 1$, as in (iii). However, for rapidly varying media which are not too thick, (32) [or in general, (25)] and (33) may give quite different results.

v) *Thin media of Type II*: The reflection coefficients for such media may differ considerably from the "good path" result, and so somewhat surprising values may occur. An example is given in Part IX, Fig. 21.

VI. Accurate Reflection Coefficients. Regardless of the accuracy or inaccuracy of the discussion in Part IV when applied to general cases, the formulae (25) and (27) can be used for specially constructed media as follows:

Use any function $\rho_1(x)$ which equals 1 for $x < -x_a$, and remains positive and has a continuous first derivative everywhere. Define

$$\rho_2(x) = \frac{1}{2k} \frac{d}{dx} \log \rho_1 \tag{34}$$

and construct a medium for which

$$n^2 = \rho_1^2 - \rho_2^2 + \frac{1}{k}\frac{d}{dx}\rho_2 \tag{35}$$

using any specified k.

For this medium, at the frequency $kc/2\pi$, we will have

$$r_1 = \rho_1 ; \qquad r_2 = \rho_2 \tag{36}$$

exactly. To satisfy the conditions (5) on n, ρ_1 would have to have a certain form for $x > x_a$. This difficulty can be avoided, however, by using another function ρ_1' which equals 1 for $x > x_a$, and remains positive and has a continuous first derivative everywhere. The medium to be used will have n^2 given by (35) for $x < x_e$ and by a similar relation in ρ_1', ρ_2' for $x > x_e$, x_e being a point where the two relations give the same value. (The existence of at least one such point is a further condition, though we may now remove the restrictions on ρ_1 to the right, and those on ρ_1' to the left, of x_e.)

We should then be able to derive the exact reflection coefficient for the medium so constructed, at the frequency $kc/2\pi$. Using the discussion of Part IV as a guide, one could obtain a specified form of medium. For media with large regions of negative n^2, it is easier to choose ρ_1 and ρ_1' over part of the range only, and choose ρ_2 and ρ_2' over the remainder ($x_b < x < x_d$). Examples are given in Part IX.

It should be pointed out that this technique is accurate in all cases. The variations may be as rapid as we please, and n^2 need not be restricted by the conditions imposed in this paper (see, for example, Fig. 26, Part IX, where $n \neq 1$ on the right). Even complex n^2's may be obtained, if proper modifications are made. There are certain disadvantages to the method, however:

i) The reflection coefficient is only known for the frequency used in the construction. For nearby frequencies we may expect little change, but we have no way of estimating the amount. In some cases (see Part VIII) the frequency is very critical in determining reflection. However, if we choose functions to give curves as in Fig. 12, we can be fairly sure that for higher frequencies the curves approach the limiting values mentioned before (aside from dispersion effects).

ii) The exact form of the medium, i.e. $n^2 = n^2(x)$, is not known until after the construction is made. However, with proper judgment, we could construct almost any medium desired, at least approximately.

iii) In general, dn^2/dx will be discontinuous at $x = x_e$. This is not a serious fault, however, and need not occur if care is taken in selecting the ρ functions.

VII. Non-Reflecting Media. One of the primary values of the method just developed is that it shows the existence of non-reflecting media (for any given frequency) even when n^2 is considerably negative over a considerable distance. To see this, we note that if ρ_1 is taken so it equals 1 for $x > x_a$, then we may take $\rho_1' = \rho_1$, hence $r_1 = r_1'$, $r_2 = r_2'$, and hence $|R| = 0$. Using the great freedom of choice of ρ_1 allowed, we can construct a great variety of media for which the reflection coefficient is zero (for the frequency used in construction). This applies to rapidly varying media as well as slowly varying ones. The possibility of non-reflecting media was noted by Rayleigh [1], based on similar reasoning, but not developed by him nor applied for cases of negative n^2.

If we choose ρ_1 so that ρ_2 is appreciable when $\rho_1 \approx 0$ we can get a medium with n^2 appreciably negative (and for as great a distance as we wish) which has the property

of non-reflection (for the frequency used). Several non-reflecting media are sketched in Figs. 13, 14, 15, and 16, the last two being of this type.

All non-reflecting media of the types considered in this paper must have

$$\int_{-x_a}^{x_a} r_2(x)\, dx = 0 \tag{37}$$

in order that $r_1(x_a) = r_1(-x_a)$. Since regions of negative n^2 usually arise due to $|r_2|$ becoming large, we can expect that a non-reflecting medium having one region of negative n^2 (due, say, to $-r_2$ becoming large) will have another one (due to $+r_2$ becoming

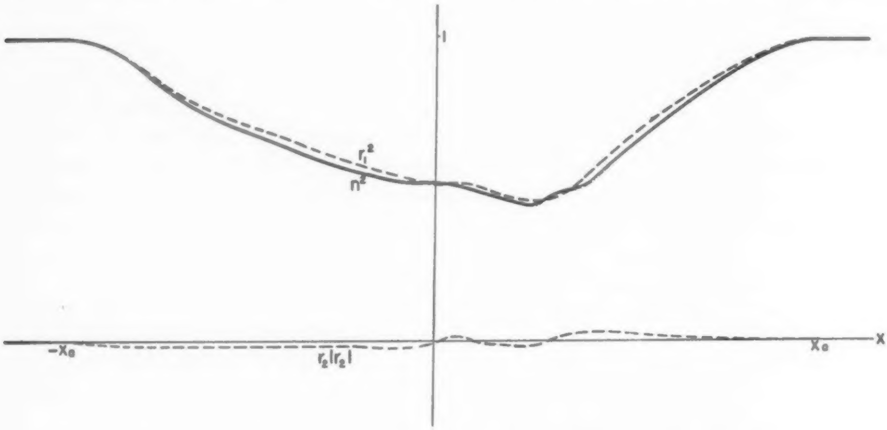

Fig. 13 A NON-REFLECTING MEDIUM (FOR A CERTAIN FREQUENCY). r_1^2 CURVE PRESCRIBED

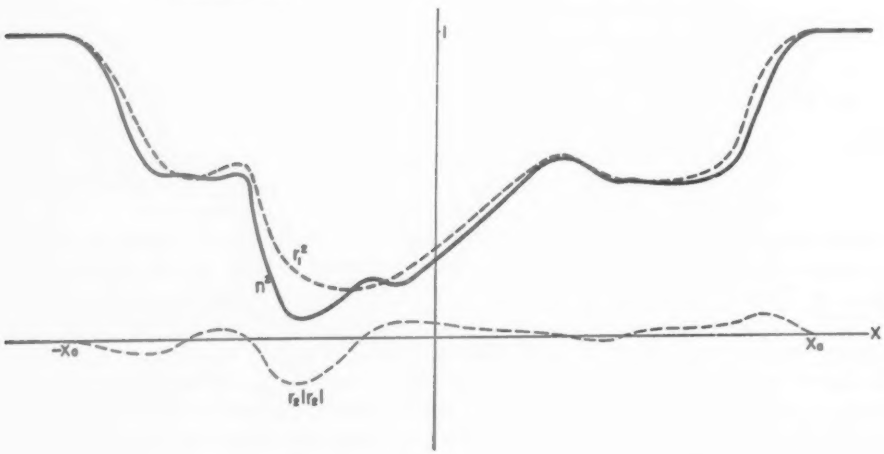

Fig. 14 A NON-REFLECTING MEDIUM (FOR A CERTAIN FREQUENCY). r_1^2 CURVE PRESCRIBED

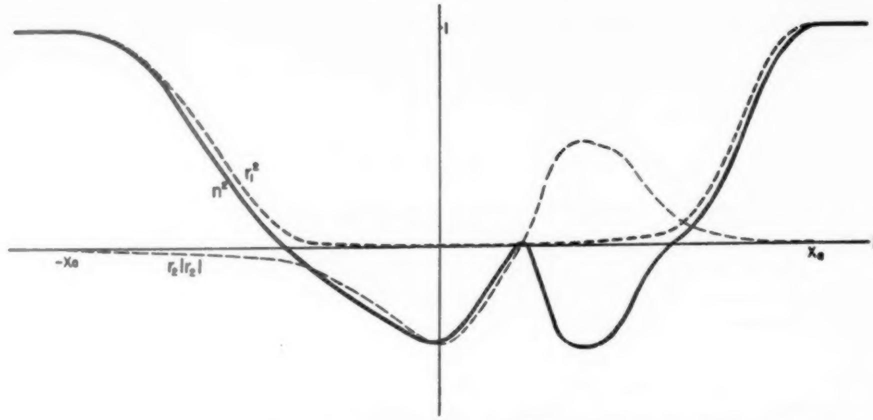

Fig. 15 A NON-REFLECTING MEDIUM (FOR A CERTAIN FREQUENCY) WITH TWO REGIONS OF NEGATIVE n^2. r_1^2 CURVE PRESCRIBED

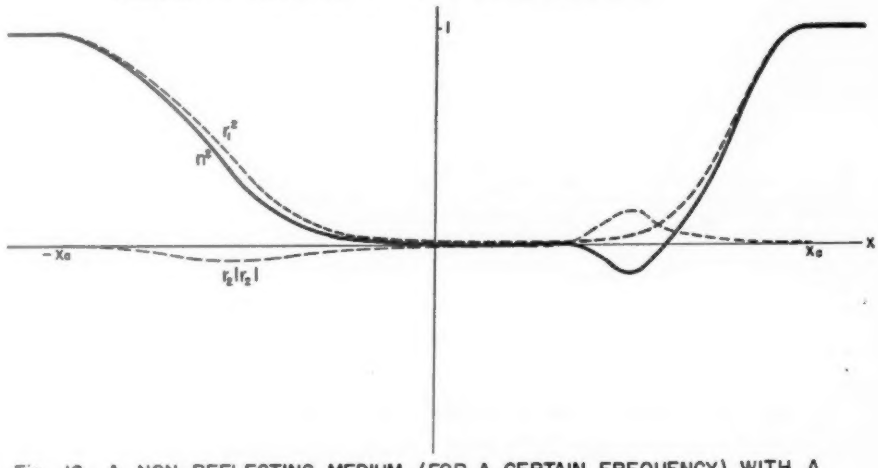

Fig. 16 A NON-REFLECTING MEDIUM (FOR A CERTAIN FREQUENCY). WITH A REGION OF NEGATIVE n^2. r_1^2 CURVE PRESCRIBED

large) or else one in which $n^2 \approx 0$ for a long distance (due to r_2 being positive but small for a long distance, r_1 growing only slowly). The two cases are illustrated by Figs. 15 and 16. In the case of very thin regions of varying n^2, in which r_1 and r_2 may vary rapidly, these results need not follow (as, for example, Fig. 21, Part IX).

For media which do not satisfy (5), equation (37) and the results derived from it need not hold in cases of non-reflection. This is illustrated in Part IX, Fig. 26.

VIII. Reflection from Media with Two Regions of Negative n^2. General media having n^2 negative in two separated regions cannot be treated by formulae (25) and (27) directly, because of the difficulty of extending r_1, r_2, r_1', and r_2' curves to give values at any one point. Useable formulae for this type of media can be obtained by the following extension of the previous methods:

Introduce two more solutions of (7)

$$p^+(x) = p_1 + ip_2 \; ; \qquad p^-(x) = -p_1 + ip_2 \tag{38}$$

where p_1 and p_2 are real, such that

$$p_1 = |\, n \,|, \qquad p_2 = dp_2/dx = 0 \qquad \text{at} \qquad x = x_c \tag{39}$$

where x_c is the point at which n^2 is a maximum (>0) between the two negative regions. Curves of p_1 and p_2 may be drawn by the same reasoning as used for r_1 and r_2, at least if $|\, n(x_c) \,|$ is not too small. This gives curves such as those shown in Figs. 17 and 18.

By introducing corresponding solutions of (1), and proceeding as before, we get the complex reflection coefficient

$$R = -\exp\left[2ik \int_{-x_a}^{x_1} r_1 \, dx\right] \frac{A \exp\,[-2ik \int_{x_1}^{x_2} p_1 \, dx] - B}{C \exp\,[-2ik \int_{x_1}^{x_2} p_1 \, dx] - D} \tag{40}$$

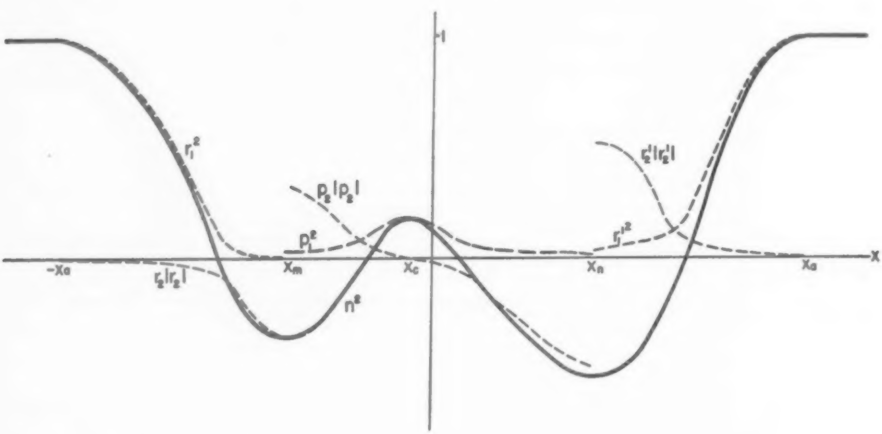

Fig. 17 CURVES FOR A MEDIUM WITH TWO REGIONS OF NEGATIVE n^2

where

$$A = (r^+ - p^+)_{x=x_1}(p^- - r')_{x=x_2}$$

$$B = (r^+ - p^-)_{x=x_1}(p^+ - r')_{x=x_2}$$

$$C = (r^- - p^+)_{x=x_1}(p^- - r')_{x=x_2}$$

$$D = (r^- - p^-)_{x=x_1}(p^+ - r')_{x=x_2}$$

and x_1 and x_2 are any two values of x. For purposes of evaluating (40) we would take x_1 near x_m, the point of minimum n^2 on the left, and x_2 near x_n, the point of minimum n^2 on the right.

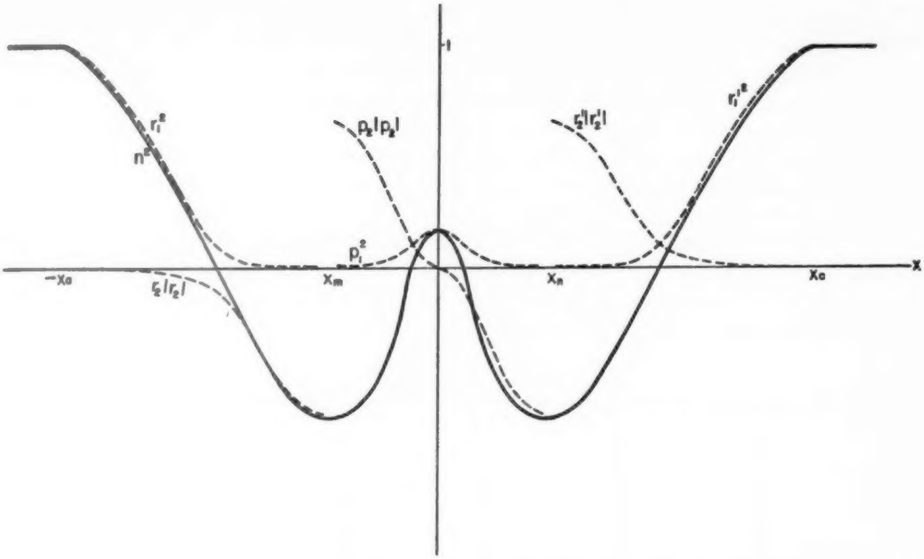

Fig. 18 CURVES FOR A SYMMETRICAL MEDIUM WITH TWO REGIONS OF NEGATIVE n^2

In slowly varying media, with sufficiently deep minima of n^2, we will have $-r_2 \gg r_1$, $p_2 \gg p_1$ at x_m, and $r_2' \gg r_1'$, $-p_2 \gg p_1$ at x_n. Taking $x_1 = x_m$, $x_2 = x_n$, as we shall from now on, we find $A \approx B \approx C \approx D$ and hence, in most cases, $|R| \approx 1$, the expected result.

However, if two conditions are fulfilled, namely

$$|A| = |B| \tag{41}$$

$$\varphi_A - 2k \int_{x_1}^{x_2} p_1 \, dx = \varphi_B - 2q\pi \tag{42}$$

then we will have, from (40), $|R| = 0$. Here φ_A = phase angle of A, φ_B = phase angle of B, and $q = 0, \pm 1, \pm 2, \cdots$.

If the medium is symmetrical, $|A| = |B|$ for all frequencies, as can be shown by choosing $x_1 = -x_2$ and comparing the quantities involved. The second condition, (42), will probably be fulfilled for a whole succession of frequencies. If the medium has curves like those of Fig. 18, where $-r_2(x_m) \gg r_1(x_m)$ etc., then $\varphi_A \approx \varphi_B \approx \pi$ nearly independently of frequency. Thus $|R|$ will vanish every time $2k \int_{x_1}^{x_2} p_1 \, dx$ increases by 2π as k varies.

It can further be shown that, for such a medium, the band width Δk of passed waves (those having $|R| < 1/\sqrt{2}$) is given by

$$\Delta k \int_{x_1}^{x_2} p_1 \, dx = \left[\frac{4 r_1 p_1}{(p_2 - r_2)^2} \right]_{x=x_1}. \tag{42}$$

On the assumptions made in developing this formula, Δk will be a very small quantity.

These media act as very narrow band pass filters, then. For such a medium, a graph of $| R |$ against $2k \int_{x_1}^{x_2} p_1 \, dx$ would appear much like that sketched in Fig. 19.

For media having the minimum of n^2 not very deep, the maxima of $| R |$ would not be so close to 1, nor would the dips in the curve be as narrow. In a plot of $| R |$ against k, some distortion of the graph will occur. This comes from the change of $p_1(x)$ with k, the changes in φ_A and φ_B with k, and a possible dispersion $n^2 = n^2(k)$.

Although these results have been derived for symmetrical media, there is no reason to believe that non-symmetrical media need behave much differently. Possibly $| R |$ would not become zero in most cases, even for selected frequencies, but it would probably become small for a sequence of frequencies, with intervening values near 1. Certainly we can construct non-symmetrical media for which there is no reflection, and in such cases we might expect a marked decrease in $| R |$ for a whole sequence of frequencies.

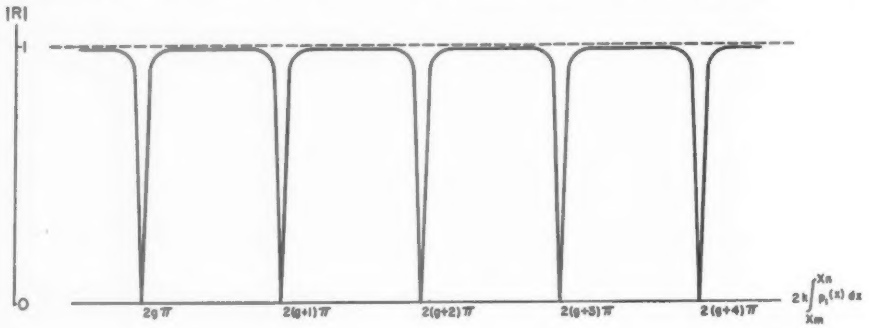

Fig. 19 REFLECTION COEFFICIENT FOR A SYMMETRICAL MEDIUM

The condition (42), and the very nature of media with two regions of negative n^2, remind one of the theory of the Fabry and Perot interferometer. This suggests as the explanation of perfect transmissions through these media a process of multiple reflection with subsequent interference. It further suggests that the phenomenon will occur to some degree even if the media are somewhat absorbing. Comparison should also be made with the problem of periodically varying media, as treated in Mathieu's equation, where a sequence of bands may be passed. (See Brillouin [15].)

IX. Accurate Reflection Coefficients in Particular Cases. Fig. 20a shows the macroscopic appearance of a symmetrical medium having minimum $n^2 = 0$. This was drawn on the basis of an assigned r_1^2 curve for $| x | > 10$, using $k = 100$ (i.e. $\lambda_{vac} = .063$ units). This figure actually shows n^2 for three media which differ almost inappreciably from one another in the region $| x | < 3$ and are identical outside this region. For $| x | < 10$, n^2 is derived by specifying r_2 (which is continuous, at $x = 10$, with the r_2 curve outside). The three media, with the corresponding r_2 curves, are shown over the range $-6 < x < 0$, on a greatly enlarged scale, in Figs. 20b, c, d. They have values of $| R |$ equal to $1/\sqrt{2}$, 0, and .92 respectively. It is evident from this that very slight changes in the form of n^2 can change the reflection coefficient remarkably, and that $1/\sqrt{2}$ is by no means a general result for these media.

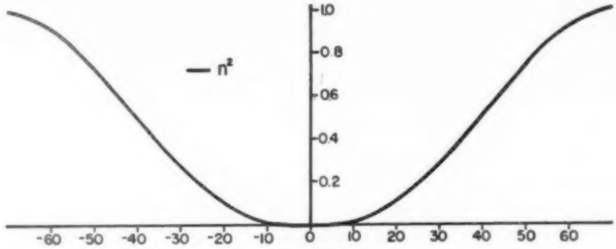

Fig. 20 a

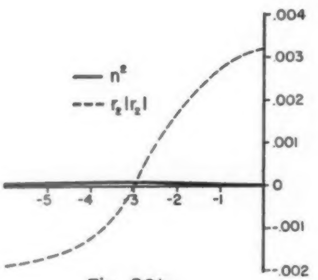

Fig. 20b

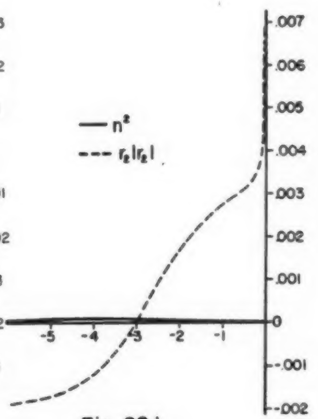

Fig. 20d

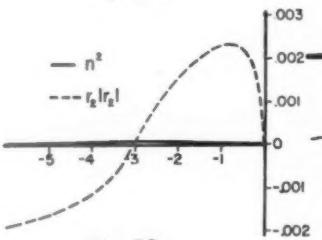

Fig. 20c

Fig. 21 shows a thin, fairly rapidly varying medium, having n^2 negative in one region only, and having zero reflection for the frequency used in construction. The curve of r_2 (not shown) was specified, and k taken as 1 ($\lambda_{vac} = 6.3$ units). In coming from the left, it can be seen that this medium is of Type II and that Dr_1 remains positive

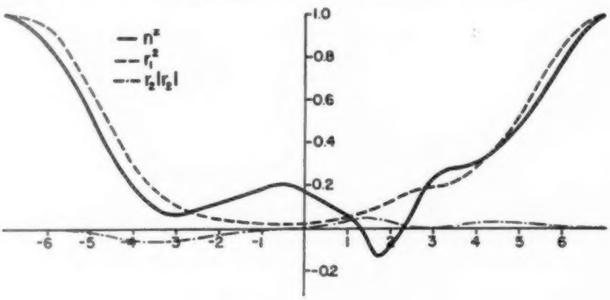

Fig. 21

throughout the region of negative n^2. In this case, $\int_{x_3}^{x_4} r_1 \, dx \approx 0.34$ which is less than π, in agreement with Eq. (31). The reflection coefficient modulus, based on the good path method, is 0.79, which is in obvious disagreement with the correct value, zero.

Fig. 22 shows, on the right, one half of a symmetrical medium, and on the left, the curves of r_1^2 and $r_2 \mid r_2 \mid$ from which it was constructed. Here $k = 10$, $\lambda_{vac} = 0.63$ units. The reflection coefficient for this wavelength has modulus which differs inappreciably from 1 when calculated by either (32) or (33). The transmission coefficient has modulus approximately equal to ϵ^{-65}; the good path method gives ϵ^{-69}.

Fig. 23, shows similar curves for another medium. Here, $k = 1$ and $\lambda_{vac} = 6.3$ units for the construction. For these waves, $\mid R \mid = 0.954$, while the good path method gives $\mid R \mid = 0.998$. The medium is, of course, rapidly varying and "narrow" relative to the

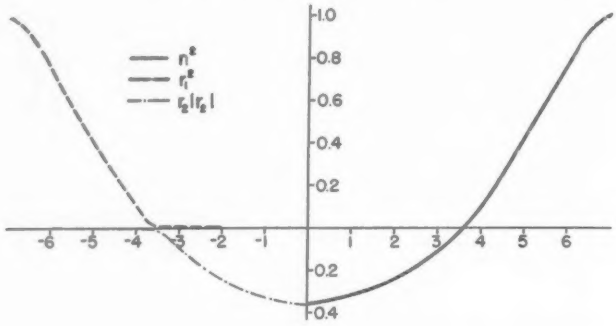

Fig. 22

wavelength used. Figs. 21 and 22 may be compared to see how closely the r curves "stick to" the n curves in the two cases, "fairly slowly" and "fairly rapidly" varying.

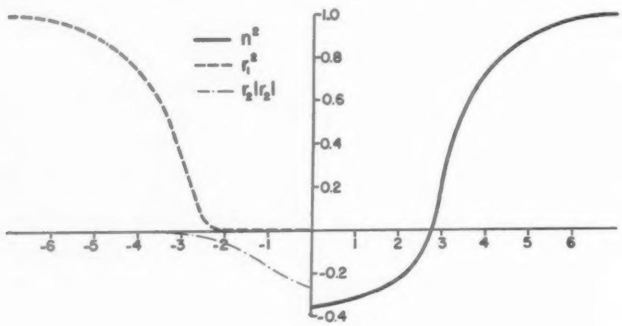

Fig. 23

Figs 24a and b show curves for a medium such as occurs in ionospheric work. The value of k used was 1000, corresponding to $\lambda_{vac} = 0.0063$ units. For an isonosphere 120 km. thick, this corresponds to $\lambda_{vac} = 63$ m, or a frequency of 4.8 mc/s. Reflection is essentially complete here. With such large values of k (for media of this thickness) there is practically no deviation of the r_1^2 and $r_2 \mid r_2 \mid$ curves from the limiting forms

mentioned previously, and almost identical curves would be obtained with any k of this or greater order of magnitude. In the case of the ionosphere, of course, the medium is dispersive and there would be a change of the n^2 curve as k is changed. Inter-modal coupling must also be included before a complete solution is obtained in this problem.

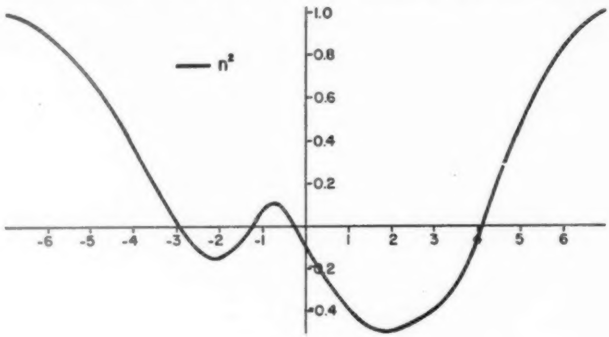

Fig. 24a

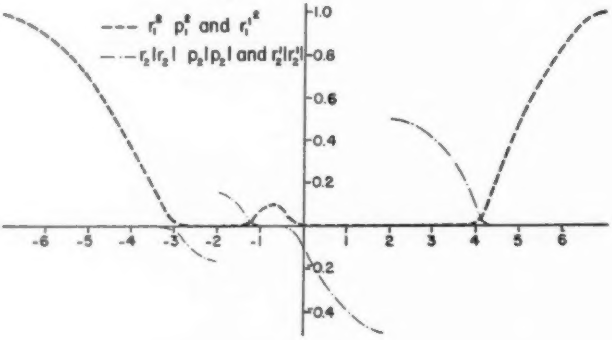

Fig. 24b

Fig. 25 shows, on the right, half of a symmetrical medium having two regions of negative n^2; on the left are curves used in deriving this medium. A value of $k = 10$

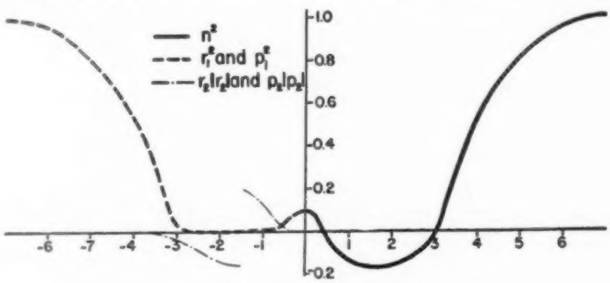

Fig. 25

was used ($\lambda_{vac} = 0.63$ units), but similar r_1^2 and $r_2 \mid r_2 \mid$ curves would be expected for any k near this. Here $\int_{x_m}^{x_a} p_1 \, dx \approx 0.36$, and at x_m, $r_1 \approx \epsilon^{-1.2k}$, $r_2 \approx 0.4$, $p_1 \approx \epsilon^{-0.7k}$, and $p_2 \approx 0.4$. These approximations will be fairly good for all k near 10. Then $\varphi_A \approx \varphi_B \approx \pi$ as stated before. There will then be no reflection of waves having $k \approx 8.7$, $17.4, \cdots$ ($\approx 2.8\, q\pi$ with $q = 1, 2, \cdots$). The band widths passed are given by $\Delta k \approx 4\epsilon^{-1.9k}/(0.8)^2(0.36)$. For $k = 8.7$ this is $\Delta k \approx 1.1 \times 10^{-6}$ and for $k = 17.4$ it is $\Delta k \approx 7 \times 10^{-14}$. These are obviously very narrow bands.

Fig. 26 shows a medium which has n^2 negative only once, and which has $n^2 = 1$ on the left and $n^2 \approx 0.003$ on the right. There is no reflection for the waves used in construction, which had $k = 1$ and $\lambda_{vac} = 6.3$ units.

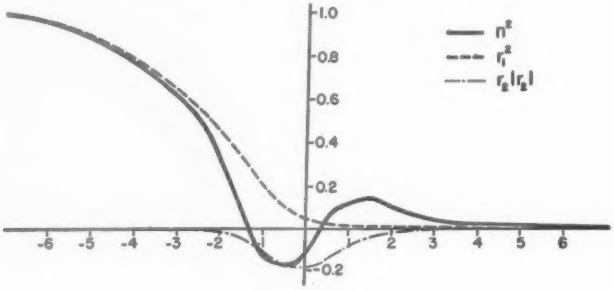

Fig. 26

X. Mass Waves. In quantum theory, an equation of the form of (1) comes from eliminating t from

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + V(x)\Psi \tag{43}$$

where $\hbar$ is Planck's constant divided by 2π, m is the mass of a particle, and $V(x)$ is a potential energy field which varies only with the x coordinate. We set

$$\Psi = f(x)\epsilon^{-iEt/\hbar} \tag{44}$$

where E is to be the (constant) total energy of the particle. This gives

$$\frac{d^2 f}{dx^2} + \frac{2m}{\hbar^2} [E - V(x)]f = 0. \tag{45}$$

The forms of (45) and (4) are identical; the solutions will be identical if we set

$$k^2 n^2 = [E - V(x)]2m/\hbar^2. \tag{46}$$

The factorization of the right hand side into k^2 and n^2 parts may be done arbitrarily, but the k^2 part must be independent of x if our previous equations are to hold. If $V = 0$ for $\mid x \mid \geq x_a$, then the most convenient association is

$$k^2 = 2mE/\hbar^2; \qquad n^2 = 1 - V(x)/E \tag{47}$$

since this gives $n^2 = 1$ for $\mid x \mid \geq x_a$ as before. The other restrictions previously placed on n^2 correspond to V being a real continuous function of x which is non-negative between $-x_a$ and x_a. This is then, the potential barrier of quantum mechanics. The cases of maximum V being less than, equal to, or greater than, E, correspond to minimum n^2 being positive, zero, or negative, respectively.

The results of all the previous parts of the paper may now be applied to the present problem. One modification only is important; we are usually interested in the behaviour of the solutions as E is varied, keeping m and V fixed. This corresponds not only to a variation in k but also to a variation in n^2—i.e., the media considered are dispersive.

Physically, the results of this paper when applied to mass waves indicate that streams of particles of the proper energy can get through some potential barriers without any reflection. In the case of double barriers, a whole spectrum of such beams could pass. It may be possible to apply this result to obtain sharp ranges of velocities for such experiments as Thomson's e/m determination.

Since the radial wave equation can be put in the same form as (45), the methods of the present paper can be applied in radial problems. For example, exact eigenstate conditions can be established, in terms of $r(x)$, for diatomic molecules. The potentials occurring in such problems do not lead to the type of n^2 curves considered in this paper, however, so the matter will not be pursued here.

XI. Remarks. The results of this paper are in general agreement with those of other methods. However, by showing that unexpected reflection coefficients may occur, they contain a warning against using the formulae of the other methods in cases where they have not been proven applicable, such as media with two regions of negative n^2, and media having roots of n^2 in the complex plane near those on the real x-axis (see [15]).

This investigation was carried out at the Radio Physics Laboratory of the Defence Research Board, Ottawa. The problem was proposed by Mr. J. C. W. Scott, whose suggestions and encouragement are much appreciated.

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CONNECTION FORMULAS BETWEEN THE SOLUTIONS OF MATHIEU'S EQUATION*

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Abstract. The problem of connecting the various types of solutions of Mathieu's equation is solved by the introduction of a new parameter Φ which is a function of the the two equation parameters a and q. This quantity Φ is introduced and enclosed between two very close analytic limits in section 2. In sections 3, 4, 5 precise definitions are given and information is collected for the three main types of functions which are to be connected. Section 6 contains the connection formulas. Section 7 reviews the status of knowledge achieved. Section 8 is an appendix on integral equations which are more general than those developed earlier in the text, but which appear to be of no use for the main purpose of this paper.

1. Introduction. A variety of different types of solutions have been written down for the Mathieu differential equation

$$\frac{d^2f}{dx^2} + (a - 2q \cos 2x)f(x) = 0 \tag{1}$$

a general solution of which we shall call *me* x. The special solutions proposed stress different qualitative features of this general solution and suggest themselves in different types of applications. By general principles, there must exist a connection formula between any three such solutions. It is the purpose of this paper to write down these connection formulas for some of the more important solutions of equation (1).

In carrying out this program we shall assume q real, and once it is assumed real it may be assumed positive because the transformation

$$x \to \frac{\pi}{2} + x$$

will reverse the sign of q. We shall express this sometime by writing

$$q = k^2. \tag{2}$$

The parameter a will also be assumed real. The treatment is particularly designed for positive a; this becomes important in the discussion of the next section.

2. On the function *ke* y. Equation (1) contains three distinct real equations, one of which is (1) itself. The second is obtained by setting

$$x = iz$$

which yields

$$\frac{d^2f}{dz^2} - (a - 2q \cosh 2z)f = 0 \tag{3}$$

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and the third by setting

$$x = \frac{\pi}{2} + iy$$

which yields

$$\frac{d^2f}{dy^2} - (a + 2q \cosh 2y)f = 0. \tag{4}$$

The study of equation (4) for f as a function of the real variable y will occupy the rest of this section.

Whenever $| y |$ is large equation (4) can be approximated closely by

$$\frac{d^2f}{dy^2} - (a + qe^{2|y|})f = 0 \tag{5}$$

which admits as solutions the modified Bessel Functions

$$f_1(y) = K_{a^{1/2}}(k\, e^{|y|}), \qquad f_2(y) = I_{a^{1/2}}(k\, e^{|y|}). \tag{6}$$

The solutions (6) give the asymptotic character of the solutions of (4); they show us that there must be one solution vanishing as $K_{a^{1/2}}(k\, e^{y})$ for positive y. We shall call it $ke\ y$; we shall normalize it by the prescription that

$$ke\ y \sim \frac{\exp(-k\, e^{y})}{(k\, e^{y})^{1/2}} \qquad \text{for} \qquad y \gg 0. \tag{7}$$

We now assume that in equation (4), we have the inequality

$$a + 2q > 0. \tag{8}$$

The solutions of (4) are then always curved away from the axis. This means for $ke\ y$ that it stays positive throughout, and that it cannot vanish exponentially for large negative y. Hence we must be able to write

$$ke\ y \sim \frac{\exp[k\, e^{|y|} + \Phi(a, k)]}{(k\, e^{|y|})^{1/2}} \qquad \text{for} \qquad y \ll 0 \tag{9}$$

with $\Phi\ (a, k)$ real.

We shall now show that under the restriction (8), $\Phi(a, k)$ is a smoothly varying function of a and k, that is, it does not partake in the oscillatory character of the parameter β introduced by Floquet's theorem. We shall establish this fact in the remainder of this section by enclosing Φ between narrow limits, neither one of which shows any oscillation. These limits are reproduced in Figs. 1 and 2; they furnish incidentally a close numerical approximation to Φ when desired. The structural discussion of Mathieu's equation will then be resumed in section 3.

We start out with the approximate value of Φ which is obtained by application of the Jeffreys' or WKB method[1] to equation (4). We set

$$ke\ y = A(y)e^{-S(y)} \tag{10}$$

[1] H. Jeffreys. Proc. Lond. Math. Soc. **23**, 428 (1924).

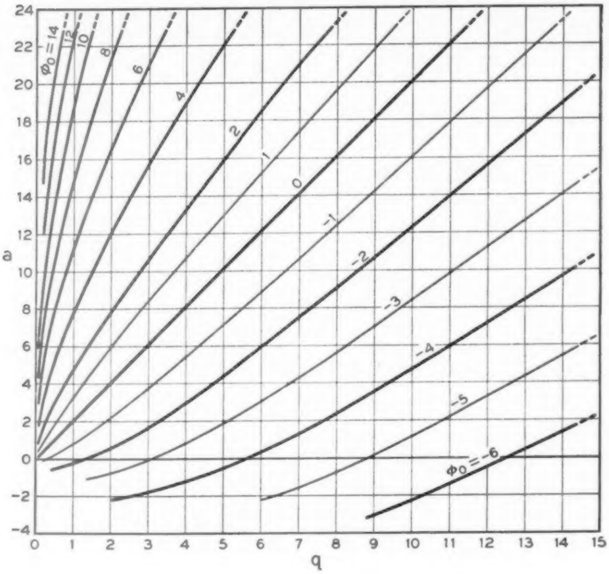

FIG. 1 MAP OF THE LOWER BOUND ϕ_0 OF $\phi(a,q)$. THIS LOWER BOUND RESULTS FROM THE JEFFREYS APPROXIMATION.

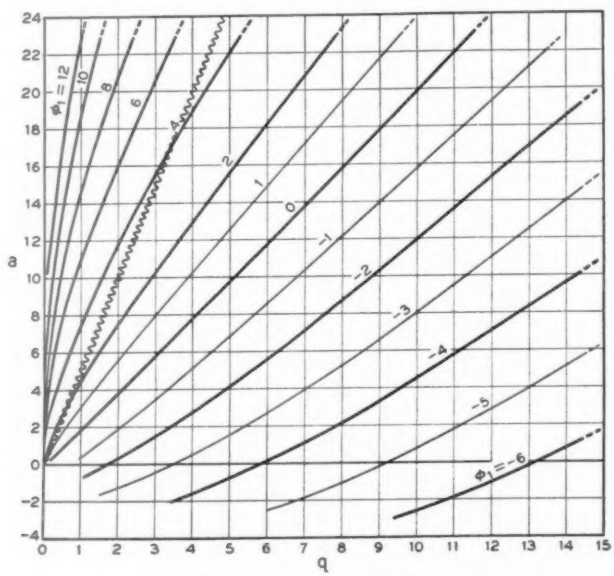

FIG. 2 UPPER BOUND ϕ_1 OF $\phi(a,q)$. THE BOUND IS NOT PROVED TO THE LEFT OF THE WAVY LINE, BUT PROBABLY HOLDS THERE ALSO.

and set

$$\left(\frac{dS}{dy}\right)^2 = a + 2q \cosh 2y. \tag{11}$$

This yields for A the equation

$$\frac{d^2 A}{dy^2} - 2 \frac{dA}{dy}\frac{dS}{dy} - A \frac{d^2 S}{dy^2} = 0. \tag{12}$$

The standard solution is obtained by neglecting the first term in (12). We find in this case

$$ke\ y \approx \frac{\exp\left[\frac{1}{2}\Phi_0 - \int_0^y (a + 2k^2 \cosh 2\eta)^{1/2}\, d\eta\right]}{(a + 2k^2 \cosh 2y)^{1/4}}. \tag{13}$$

The term $\frac{1}{2}\Phi_0$ enters into the exponent in order to satisfy the asymptotic requirement (7); it obeys the relation

$$\frac{1}{2}\Phi_0 = \lim_{y=\infty}\left[\int_0^y (a + 2k^2 \cosh 2\eta)^{1/2}\, d\eta - 2k \sinh y\right]. \tag{14}$$

Equation (14) then represents the Jeffreys approximation to Φ as defined in (9):

$$\Phi \approx \Phi_0\ . \tag{15}$$

Φ_0 is easily evaluated in terms of complete elliptic integrals; we find

for $a \geqq 2k^2$

$$\Phi_0 = 2\,\frac{a - 2k^2}{(a + 2k^2)^{1/2}}\, D\left[\left(\frac{a - 2k^2}{a + 2k^2}\right)^{1/2}\right], \tag{16a}$$

for $a \leqq 2k^2$

$$\Phi_0 = -\frac{2k^2 - a}{k}\left\{K\left[\frac{(2k^2 - a)^{1/2}}{2k}\right] - D\left[\frac{(2k^2 - a)^{1/2}}{2k}\right]\right\}. \tag{16b}$$

The two expressions (16) do not indicate a break at $a = 2k^2$, for they are analytic continuations of each other.

The approximation (16) to Φ is shown in Fig. 1; we shall now prove for it that the approximate identity (15) is an inequality, that is, that

$$\Phi > \Phi_0\ . \tag{17}$$

To prove this let us write equation (12) in the form

$$A^2 \frac{d^2 S}{dy^2} + 2A \frac{dA}{dy}\frac{dS}{dy} - A \frac{d^2 A}{dy^2} = 0.$$

Subtracting $(dA/dy)^2$ on either side we get

$$\frac{d}{dy}\left(A^2 \frac{dS}{dy} - A \frac{dA}{dy}\right) = -\left(\frac{dA}{dy}\right)^2 < 0.$$

Integrating this inequality between $-\infty$ and $+\infty$, and observing that $A(dA/dy)$ vanishes at either end, we get

$$\left(A^2 \frac{dS}{dy}\right)_{y\to+\infty} - \left(A^2 \frac{dS}{dy}\right)_{y\to-\infty} < 0.$$

If the Jeffreys' approximation were correct this difference would be zero; actually, the formulas (7), (9), (10), (11) and (14) yield

$$A(+\infty) \sim \frac{\exp\{\frac{1}{2}\Phi_0\}}{(dS/dy)^{1/2}}, \tag{18a}$$

$$A(-\infty) \sim \frac{\exp\{-\frac{1}{2}\Phi_0 + \Phi\}}{(dS/dy)^{1/2}}. \tag{18b}$$

These expressions reduce the inequality to

$$e^{\Phi_0} < e^{-\Phi_0 + 2\Phi}$$

which is equivalent to (17).

In order to gain an upper limit for Φ, we transform equation (12) by the substitution

$$\frac{1}{A}\frac{dA}{dy} = B \tag{19}$$

which yields the equation

$$\frac{d^2S}{dy^2} + 2B\frac{dS}{dy} = \frac{dB}{dy} + B^2,$$

and hence the inequality

$$\frac{dB}{dy} - 2B\frac{dS}{dy} - \frac{d^2S}{dy^2} < 0.$$

Multiplying with e^{-2S} and integrating we get from this

$$e^{-2S}B + \int_y^\infty e^{-2S}\frac{d^2S}{dy^2}\,dy > 0.$$

Returning to A by (19) we get

$$\frac{1}{A}\frac{dA}{dy} + e^{2S}\int_y^\infty e^{-2S}\frac{d^2S}{dy^2}\,dy > 0.$$

Integrating the second term by parts we get this in the form

$$\frac{d}{dy}\ln\left[A\left(\frac{dS}{dy}\right)^{1/2}\right] + \frac{1}{2}e^{2S}\int_y^\infty e^{-2S}\frac{d}{dy}\left(\frac{d^2S/dy^2}{dS/dy}\right)dy > 0.$$

With the help of the equations (18) this is finally transformed into

$$\Phi - \Phi_0 < \frac{1}{2}\int_{-\infty}^{+\infty} e^{2S(y)}\,dy\int_y^\infty e^{-2S(\eta)}\frac{d}{d\eta}\left(\frac{d^2S/d\eta^2}{dS/d\eta}\right)d\eta. \tag{20}$$

The inequality (20) is the desired upper limit and could be evaluated by numerical methods. However, we shall proceed instead to majorize the double integral by explicit analytic expressions.

In discussing (20) we observe first that the expression in the integrand

$$\frac{d}{d\eta}\left(\frac{d^2S/d\eta^2}{dS/d\eta}\right) = \frac{4q(a\cosh 2\eta + 2q)}{(a + 2q\cosh 2\eta)^2} \tag{21}$$

is positive everywhere and vanishes strongly at either infinity; the inner integral is therefore majorized by replacing $e^{-2S(\eta)}$ by its largest value $e^{-2S(y)}$. This procedure is of no use at this stage as it gives a divergent result. However, if we integrate the outer integral by parts according to the scheme

$$\int e^{2S(y)}\,dy \sim \int d(e^{2S(y)})\,\frac{1}{2(dS/dy)}$$

then we verify by this method that the integrated out part vanishes. The expression (20) then takes the form

$$\Phi - \Phi_0 < \frac{1}{4}\int_{-\infty}^{+\infty} \frac{1}{dS/dy}\frac{d}{dy}\left(\frac{d^2S/dy^2}{dS/dy}\right)dy$$
$$+ \frac{1}{4}\int_{-\infty}^{+\infty} \frac{d^2S/dy^2}{(dS/dy)^2}\,e^{2S(y)}\,dy \int_y^{\infty} e^{-2S(\eta)}\frac{d}{d\eta}\left(\frac{d^2S/d\eta^2}{dS/d\eta}\right)d\eta.$$

The first integral is positive, as can be seen by another integration by parts:

$$\Phi - \Phi_0 < \frac{1}{4}\int_{-\infty}^{+\infty} \frac{d^2S/dy^2}{(dS/dy)^3}\,dy$$
$$+ \frac{1}{4}\int_{-\infty}^{+\infty} \frac{d^2S/dy^2}{(dS/dy)^2}\,e^{2S(y)}\,dy \int_y^{\infty} e^{-2S(\eta)}\frac{d}{d\eta}\left(\frac{d^2S/d\eta^2}{dS/d\eta}\right)d\eta. \tag{22}$$

The sign of the second integral is not immediately obvious, because the integrand changes its sign with d^2S/dy^2. An easy way to obtain an upper limit of Φ is to replace the integrand by 0 in the range in which it is negative; we find thus

$$\Phi - \Phi_0 < \frac{1}{4}\int_{-\infty}^{+\infty} \frac{(d^2S/dy^2)^2}{(dS/dy)^3}\,dy + \frac{1}{4}\int_0^{\infty} \frac{d^2S/dy^2}{(dS/dy)^2}\,e^{2S(y)}\,dy \int_y^{\infty} e^{-2S(\eta)}\frac{d}{d\eta}\left(\frac{d^2S/d\eta^2}{dS/d\eta}\right)d\eta.$$

Having now the integrand positive throughout, we can majorize it by the trick discussed above of replacing the exponent $S(\eta)$ by $S(y)$. Observing that

$$\lim_{\eta\to\infty}\frac{d^2S/d\eta^2}{dS/d\eta} = 1,$$

we get

$$\Phi - \Phi_0 < \frac{1}{4}\int_{-\infty}^{+\infty} \frac{d^2S/dy^2}{(dS/dy)^3}\,dy + \frac{1}{4}\int_0^{+\infty} \frac{d^2S/dy^2}{(dS/dy)^2}\left[1 - \frac{d^2S/dy^2}{dS/dy}\right]dy$$
$$= \frac{1}{4}\int_0^{\infty} \frac{(d^2S/dy^2)^2}{(dS/dy)^3}\,dy + \frac{1}{4}\,\frac{1}{(dS/dy)_{y=0}}\,.$$

By the observation that

$$\left| \frac{d^2S}{dy^2} \right| < \left| \frac{dS}{dy} \right|$$

the first term is seen to be smaller than the second. We get thus with (11) the inequality

$$\Phi < \Phi_2 \tag{23}$$

where

$$\Phi_2 = \Phi_0 + \frac{1}{2(a + 2q)^{1/2}}. \tag{24}$$

The upper limit (24) is shown in Fig. 3. This figure makes the limit appear rather close,

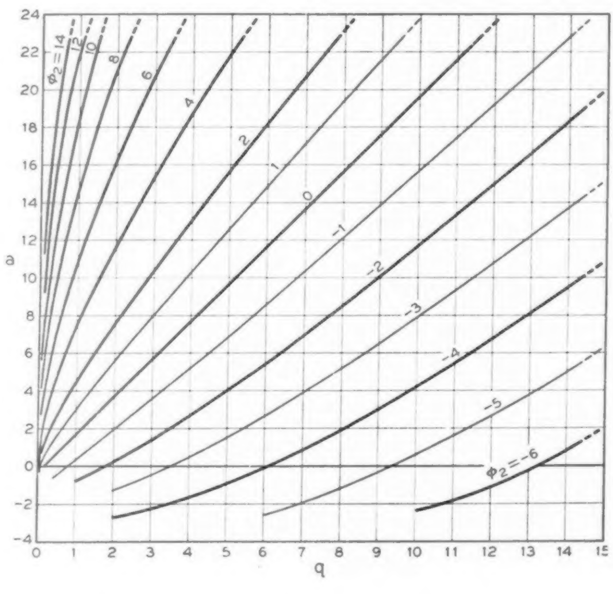

FIG. 3 UPPER BOUND ϕ_2 OF $\phi(a, q)$. THIS BOUND IS EASILY PROVED FOR ALL VALUES OF a AND q.

but a detailed study for small a and q shows deviations. We shall now establish a tighter limit by showing that the second integral in (22) is always negative and can be discarded, provided we introduce the restriction

$$a < 5q. \tag{25}$$

Introduce the abbreviation

$$E(y) = e^{2S(y)} \int_y^\infty e^{-2S(\eta)} \frac{d}{d\eta} \left(\frac{d^2S/d\eta^2}{dS/d\eta} \right) d\eta.$$

Because of (21), this function is positive everywhere and vanishes at $\pm\infty$. It is seen to obey the differential equation

$$\frac{dE}{dy} = 2\frac{dS}{dy}(E - E_{\text{lim}})$$

where

$$E_{\text{lim}} = \frac{1}{2(dS/dy)}\frac{d}{dy}\left(\frac{d^2S/dy^2}{dS/dy}\right).$$

In an E,y-plane as shown in Fig. 4, the differential equation above defines a slope at

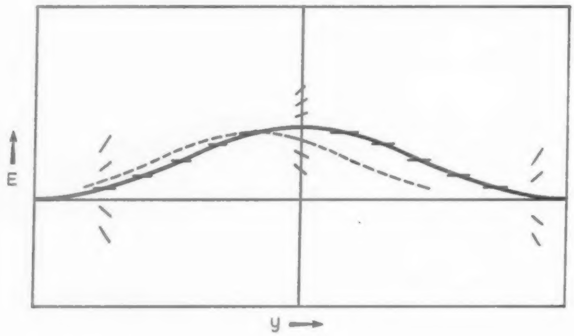

FIG. 4 DIAGRAM FOR THE STUDY OF THE AUXILIARY FUNCTION $E(y)$. THE SHORT LINES INDICATE SLOPES, THE SOLID CURVE CONNECTS POINTS OF ZERO SLOPE. THE CURVE $E(y)$ IS SHOWN DASHED. BY CONSTRUCTION, ITS VALUE FOR A POSITIVE y IS SMALLER THAN IT IS FOR $-y$.

at every point. The curve $E = E_{\text{lim}}$ divides this plane in two parts. Above this curve, the slopes are positive, below negative. The curve $E = E_{\text{lim}}$ is always crossed with zero slope. The inequality (25) is needed at this point because if it is satisfied then we find from (21) that E_{lim} has a slope whose sign is opposite to that of y. When (25) does not hold then E develops two humps as shown in Fig. 5. In the case of Fig. 4, E starts out by being 0 for $+\infty$, rises for positive decreasing values of y, but must stay below the line $E = E_{\text{lim}}$ because of the slope requirement; thus E still rises as y becomes negative. At some negative y, E reaches its maximum as it crosses the line $E = E_{\text{lim}}$; then this same slope requirement forces E to stay above the latter curve while going to zero. The result of this behavior is that we have for all y

$$0 < E(-|y|) < E(+|y|).$$

The second integral in (22) now takes the form

$$\frac{1}{4}\int_{-\infty}^{+\infty}\frac{d^2S/dy^2}{(dS/dy)^2}E(y)\,dy$$

or, for symmetry reasons

$$\frac{1}{8}\int_{-\infty}^{+\infty}\frac{d^2S/dy^2}{(dS/dy)^2}\{E(y) - E(-y)\}\,dy.$$

This last expression is negative because the curly bracket and d^2S/dy^2 have opposite sign, and thus the second integral in (22) is proved to be negative.

The proof given breaks down if the inequality (25) is reversed. The reason for this is shown on Fig. 5; as soon as E_{1im} develops a minimum at $y = 0$ instead of a maximum

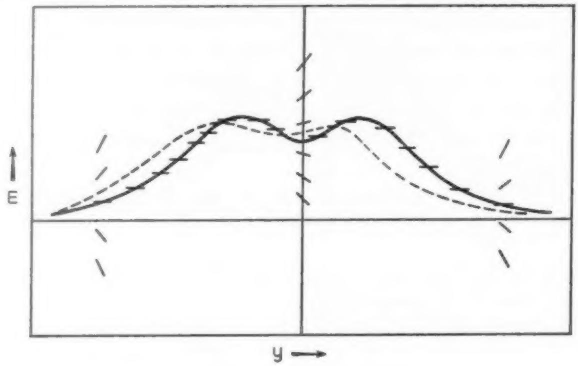

FIG. 5 DIAGRAM ANALOGOUS TO FIG. 4 FOR THE CASE WHEN INEQUALITY (24) IS REVERSED. THE FUNCTION $E(y)$ IS SHOWN IN DASHED OUTLINE. ITS MODE OF CONSTRUCTION NO LONGER IMPLIES THAT $E(|y|) \geqq E(-|y|)$.

the method of constructing E outlined above may give E a positive slope for small $|y|$. Thus the needed inequality does not hold everywhere. Nevertheless it still does hold in the greater part of the interval. Numerical checks actually indicate that the second integral in (22) is always negative.

We have thus found, at least under the restriction (25), and probably everywhere, that

$$\Phi < \Phi_1 \tag{26}$$

where

$$\Phi_1 = \Phi_0 + \frac{1}{4}\int_{-\infty}^{+\infty} \frac{(d^2S/dy^2)^2}{(dS/dy)^3}\,dy,$$

or explicitly

for $a \geqq 2k^2$

$$\Phi_1 = \left(2\frac{a - 2k^2}{(a + 2k^2)^{1/2}} - \frac{a}{3(a + 2k^2)^{3/2}}\right)D\left[\left(\frac{a - 2k^2}{a + 2k^2}\right)^{1/2}\right] + \frac{1}{3}\frac{1}{(a + 2k^2)^{1/2}}K\left[\left(\frac{a - 2k^2}{a + 2k^2}\right)^{1/2}\right], \tag{27a}$$

for $a \leqq 2k^2$

$$\Phi_1 = \left(-\frac{2k^2 - a}{k} + \frac{k}{3(2k^2 + a)}\right)K\left[\frac{(2k^2 - a)^{1/2}}{2k}\right] + \left(\frac{2k^2 - a}{k} + \frac{a}{6k(2k^2 + a)}\right)D\left[\frac{(2k^2 - a)^{1/2}}{2k}\right]. \tag{27b}$$

This upper limit is shown in Fig. 2, together with the line (25) above which the theorem (26) has not been proved. These established limits determine fairly closely the behavior of $\Phi(a, k^2)$.

It is possible without difficulty to come closer to the numerical value of Φ than Figs. 1 and 2 do. For this purpose, equation (12), which is linear in A, may be solved in successive approximations beginning with

$$2\frac{dA_0}{dy}\frac{dS}{dy} + A_0\frac{d^2S}{dy^2} = 0$$

which yields (13); successive additive corrections A_1, A_2, A_3, $\cdots$ are then obtained from the recursion system

$$2\frac{dA_n}{dy}\frac{dS}{dy} + A_n\frac{d^2S}{dy^2} = \frac{d^2A_{n-1}}{dy^2}.$$

If this calculation is carried out up to A_1, and if we take the corrected A in the form $A_0 \exp A_1/A_0$ rather than in the more obvious form $A_0 + A_1$, then a corrected Φ results which is exactly the arithmetic mean of the two limits Φ_0 and Φ_1:

$$\Phi \approx \tfrac{1}{2}(\Phi_0 + \Phi_1). \tag{28}$$

Thus a good approximation to the value of Φ is obtained by taking the arithmetic mean of the readings on Fig. 1 and Fig. 2.

3. The Lindemann-Stieltjes Functions. The following three sections define and discuss the solutions of (1) between which connection formulas are to be established. These discussions contain a good deal of information which is already available but which has to be combined with the new material to yield the desired results.

The Lindemann-Stieltjes functions[2] are based on the symmetry of the equation (1) about the points $x = 0$ and $x = \pi/2$ which are regular points of the equation. There must thus exist an even and an odd power series solution about either one of these points. We introduce the following definitions

I. *ce* $(x; a, q)$ shall be the even function about $x = 0$; its value at $x = 0$ shall be 1.
II. *se* x shall be the odd function about $x = 0$; its derivative at $x = 0$ shall be 1.
III. *de* x shall be the even function about $x = \pi/2$; its value there shall be 1.
IV. *te* x shall be the odd function about $x = \pi/2$; its derivative there shall be -1.

ce and *se* always form a linearly independent pair, as do *de* and *te*. Whenever the periodic Mathieu functions $ce_n x$ or $se_n x$ exist they are respectively identical with the generalized functions *ce* x and *se* x defined here. Similarly, ce_{2n} or se_{2n+1}, when existing, are identical with *de* x, ce_{2n+1} or se_{2n} with *te* x. The simplest realization of these functions is by the power series method:

ce x has a series in even powers of sin x.
se x " " " in odd " " "
de x " " " " even " " cos x.
te x " " " " odd " " "

[2] E. T. Whittaker and G. N. Watson. *A course of modern analysis.* Cambridge University Press, Fourth Edition, Section 19.5.

The coefficients are obtained directly from (1) through term by term recursion with the first coefficient equal 1. However, the radius of convergence of these series is only 1, and this excludes the possibility of getting connection formulas by this procedure.

As these functions are constructed relatively easily from others not having their symmetry we will not investigate their structure further, but get it indirectly from the study of the other types.

4. The Mathieu analogues to the Hankel functions. We will now use the function $ke\,y$ of section 2 as an auxiliary function to define solutions of (1) having given asymptotic character. We define four such functions, namely $he^{(1)}x$, $he^{(2)}x$, $he^{(3)}x$, $he^{(4)}x$, by the supplementary prescriptions

$$he^{(1)}\left(\frac{\pi}{2} + iy\right) = ke\,y, \tag{29}$$

$$he^{(2)}\left(-\frac{\pi}{2} + iy\right) = ke\,y, \tag{30}$$

$$he^{(3)}\left(-\frac{\pi}{2} - iy\right) = ke\,y, \tag{31}$$

$$he^{(4)}\left(\frac{\pi}{2} - iy\right) = ke\,y, \tag{32}$$

y being taken as real initially. We now drop this restriction and continue the functions outside their defining lines. As equation (1) is free of singularities for finite x, equations (29)-(32) hold then everywhere and the four functions $he^{(i)}x$ are related to each other by symmetry operations of equation (1).

The argument used to establish (6) can be repeated along any line parallel to the imaginary axis. As soon as we are sufficiently far away from the real axis, the general solution $me\,x$ will behave as

$$me\,x \sim \frac{\exp\,[\pm 2ik\,\cos x]}{[2k\,\cos x]^{1/2}}. \tag{33}$$

However, a particular asymptotic form

$$\frac{1}{[2k\,\cos x]^{1/2}}\,(A\,\exp\,[2ik\,\cos x] + B\,\exp\,[-2ik\,\cos x])$$

can define a given function only within a strip of limited width parallel to the imaginary axis. One way to see this is by observing that the above expression is formally periodic in x with period 4π, while by Floquet's theorem equation (1) has generally no such solution.

In order to obtain this range we start out by proving the integral relation

$$\int_{-i\infty}^{+i\infty} \exp\,[-2k\sinh y\,\cos u]me\,u\,du = A\,ke\,y \tag{34}$$

where $me\,u$ is an *arbitrary* solution of (1), $ke\,y$ the *special* solution of (4) defined by

(7), and A is a number whose value depends on the choice of $me\ u$. As integrals of the type (34) have been discussed in the literature[3] we need not dwell upon the formal steps necessary to prove (34); we have only to find out for what range of y this particular combination of limits and functions is chosen correctly. From the asymptotic formula (33), it follows that the integral exists as long as y is real and not negative. However, in the formal steps necessary to prove that the integral obeys equation (4), factors such as $\cosh y \cdot \cos u$ appear which demand that the exponential produce convergence; hence the integral (34) defines some solution of (4) only as long as $y > 0$. Our next observation is that the solution is always the same regardless of the choice of $me\ u$. This is so because, for symmetry reasons, we have that

$$\int_{-i\infty}^{+i\infty} \exp\left[-2k \sinh y \cos u\right] se\ u\ du = 0.$$

Thus, only one linearly independent solution of (1) is left in (34), producing always the same solution of (4) on the right hand side. That this function is just $ke\ y$ is seen by evaluating (34) for large y by the saddle point method. The saddle point is at the origin, which permits us to write

$$\exp\left[-2k \sinh y \cos i\, v\right] \sim \exp\left[-2k \sinh y(1 + \tfrac{1}{2}v^2)\right].$$

Hence

$$\int_{-i\infty}^{+i\infty} \exp\left[-2k \sinh y \cos u\right] me\ u\ du$$

$$\sim i\ me(0) \exp\left[-2k \sinh y\right] \int_{-\infty}^{+\infty} \exp\left[-k\, v^2 \sinh y\right] dv$$

$$= i\pi^{1/2}\ me(0)\, \frac{\exp\left[-2k \sinh y\right]}{(k \sinh y)^{1/2}}.$$

This asymptotic behavior is the same as the one of the definition (7) and hence equation (34) is proved for positive y.

We now use equation (34) to continue the function $ke\ y$ analytically outside this original range of definition. We start out by permitting values of y slightly off the positive real axis. As long as this deviation x is less than $\pi/2$ this goes without difficulty; for the convergence producing factor in (34) is changed from its previous value to

$$\exp\left[-2k \sinh y \cos x \cos u\right]$$

which will produce convergence as long as $\cos x > 0$ and $\sinh y > 0$. Using the definition (29) we arrive thus at the equation

$$\int_{-i\infty}^{+i\infty} \exp\left[-2ik \cos x \cos u\right] me\ u\ du = A\ he^{(1)}\ x \tag{35a}$$

which holds as long as

$$\mathcal{I}(x) > 0, \tag{35b}$$

$$0 < \mathcal{R}(x) < \pi. \tag{35c}$$

[3] N .W. McLachlan. *Theory and Application of Mathieu Functions.* Clarendon Press, 1947, Chapter X.

This range of definition can be extended further if we allow the path of integration to be deformed. The imaginary axis along which the above integral is running is one of a series of valleys along which it could be taken to produce a convergent result. The other valleys are separated from this one by distances 2π, 4π, etc., as shown in Fig. 6. These valleys have a width π, as shown by (35c); they are separated by ridges of the same width for which the integral diverges. Now as $\mathcal{R}(x)$ changes the valleys of $\exp [-2ik \cos x \cos u] = \mathcal{K}(x, u)$ shift. Set $x \to x + iy$, $u \to u + iv$ then we get for the real part of the exponent in $\mathcal{K}(x, u)$

$$\exp [-2k (\cos x \cosh y \sin u \sinh v + \sin x \sinh y \cos u \cosh v)]$$

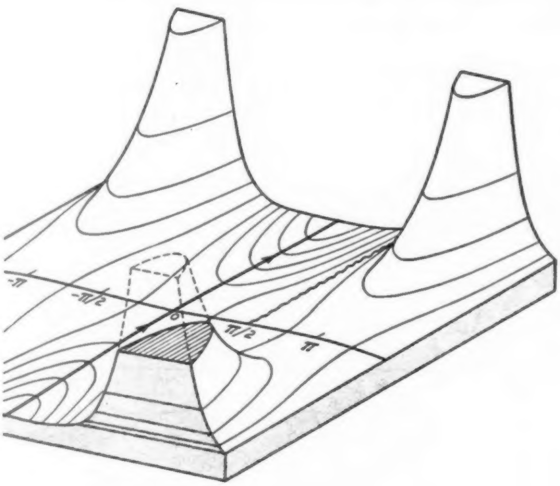

FIG. 6 PERSPECTIVE DRAWING IN THE U-PLANE SHOWING THE ABSOLUTE VALUE OF THE INTEGRAND AND THE PATH OF INTEGRATION FOR THE INTEGRAL (35). THE FACTOR ME u IS NOT INCLUDED AND x MUST LIE ON THE WAVY LINE NOT FAR FROM THE REAL AXIS

which, for v positive and large, becomes approximately

$$\exp [-ke^v (\cos x \cosh y \sin u + \sin x \sinh y \cos u)].$$

Clearly, if y is also large, the exponent contains $\sin (x + u)$ which is kept at its maximum value by setting

$$x + u = \frac{\pi}{2}. \tag{36a}$$

If y is not large then the bottom of the valley is given by

$$\tanh y \tan x \tan u = 1. \tag{36b}$$

The movement of x as a function of u does not differ essentially between (36a) and (36b). As $\mathcal{R}(x)$ decreases from $\pi/2$, $\mathcal{R}(u)$ increases by an amount which is essentially

equal to this decrease. The movement becomes gradually more jerky as y becomes smaller and ceases to function for $y = 0$. Similarly for v negative and large we find for y large

$$x - u = \frac{\pi}{2}, \tag{37a}$$

and for general y

$$\tanh y \tan x \tan u = -1. \tag{37b}$$

The movement of the valleys is thus in the opposite directions on the two sides of the real axis. This means that for decreasing $\mathcal{R}(x)$ the path of the integral (35) deforms as shown in Fig. 7.

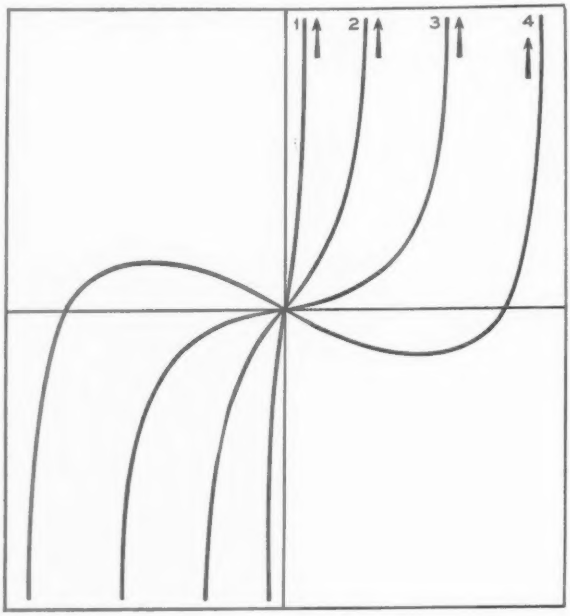

FIG. 7 DEFORMATION OF THE PATH IN THE u-PLANE FOR THE INTEGRAL (35). SUCCESSIVE NUMBERS ARE FOR R(x) DECREASING, STARTING FROM $\frac{\pi}{2}$.

We have thus found, for the entire half plane for which $\mathcal{I}(x) > 0$, the solution of (1) which has, along the line parallel to the imaginary axis and passing through $\pi/2$, the behavior prescribed by (29) and (7); the solution is given to us in the form of an integral.

$$A\, he^{(1)}x = \int_{u_1 - i\infty}^{u_2 + i\infty} \exp\,[-2ik \cos x \cos u]\, me\, u\, du \tag{38a}$$

where

$$u_1 \text{ obeys (37)}, \tag{38b}$$

$$u_2 \text{ obeys (36)}, \tag{38c}$$

$$\mathcal{I}(x) > 0. \tag{38d}$$

Fig. 8 illustrates the path for the special case $\mathcal{R}(x) = 0$.

Formula (38) will now be used to find the range to which the asymptotic formula (7) is applicable. If we decrease $\mathcal{R}(x)$ from $\pi/2$ to some smaller value the two ends of the path displace as shown in Fig. 7; this displacement leaves undisturbed the location $u = 0$ of the saddle point while at the same rotating its orientation clockwise; this rotation

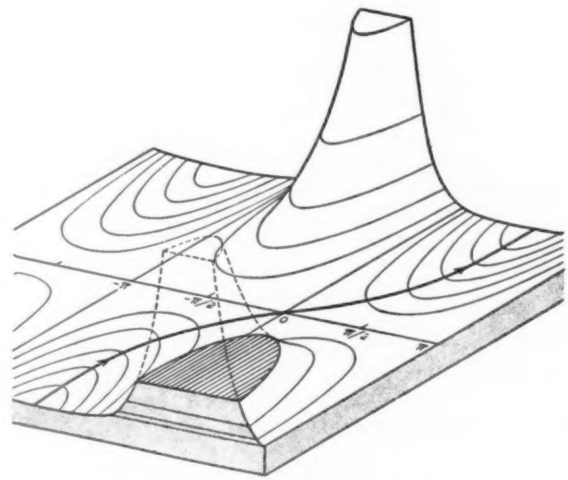

FIG. 8 PATH FOLLOWED IN THE U-PLANE BY THE INTEGRAL (38); THE CASE SHOWN IS WHEN X LIES ON THE POSITIVE IMAGINARY AXIS; THE SADDLE POINT OF THE INTEGRAL LIES AT U = 0.

shows up in the passage from Fig. 6 over Fig. 8 to Fig. 9. For large $\mathcal{I}(x)$, the angle α of the saddle with the x axis is found to be

$$\alpha = \frac{\pi}{4} + \frac{1}{2}\mathcal{R}(x) \tag{39}$$

The integral (38) yields then

$$he^{(1)}x \sim \frac{\exp\,[-2ik\cos x - i\pi/4]}{(2k\cos x)^{1/2}} \tag{40a}$$

with the restriction

$$\mathcal{I}(x) \gg 0 \tag{40b}$$

However, when $\mathcal{R}(x)$ drops below $-\pi/2$ the situation is altered. The two valleys are now so far removed that the path has to proceed over three saddles, lying at $-\pi$, 0, π. This situation is illustrated in Fig. 9. Of the three contributions, the one at the origin retains its analytical form (40a) but the two others will add to it and thus invalidate it. A similar modification must be applied when $\mathcal{R}(x)$ increases from $+\pi/2$ beyond $+3/2\pi$. The asymptotic expansion (40a) is therefore valid within the range

$$-\frac{1}{2}\pi \leqq \mathcal{R}(x) \leqq \frac{3}{2}\pi. \tag{40c}$$

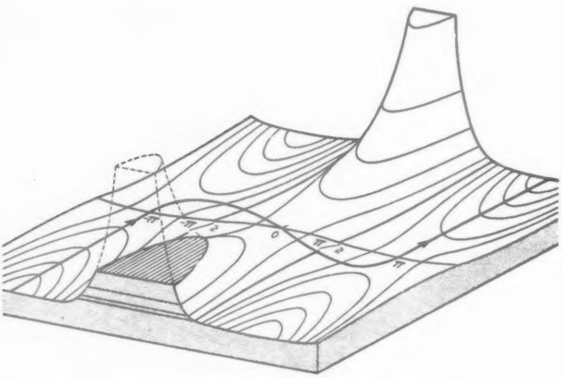

FIG. 9 PATH FOLLOWED IN THE U-PLANE BY THE INTEGRAL (38) AFTER R(x) HAS DROPPED SOMEWHAT BELOW $-\frac{\pi}{2}$. THE ORIGINAL SADDLE POINT REMAINS AT THE ORIGIN, ROTATING CLOCKWISE; IN ADDITION, TWO NEW SADDLE POINTS, AT $-\pi$ AND $+\pi$ MAKE THEIR APPEARANCE.

In the same way we obtain from equation (30)

$$he^{(2)}x \sim \frac{\exp\,[+2ik\,\cos x + i\pi/4]}{(2k\,\cos x)^{1/2}} \tag{41a}$$

if

$$-\frac{3}{2}\pi \leqq \mathcal{R}(x) \leqq +\frac{1}{2}\pi, \tag{41b}$$

$$\mathcal{I}(x) \gg 0, \tag{41c}$$

and from (31)

$$he^{(3)}x \sim \frac{\exp\,[-2ik\,\cos x - i\pi/4]}{(2k\,\cos x)^{1/2}} \tag{42a}$$

if

$$-\frac{3}{2}\pi \leqq \mathcal{R}(x) \leqq +\frac{1}{2}\pi, \tag{42b}$$

$$\mathcal{I}(x) \ll 0, \tag{42c}$$

and finally from (32)

$$he^{(4)}x \sim \frac{\exp\,[+2ik\,\cos x + i\pi/4]}{(2k\,\cos x)^{1/2}} \tag{43a}$$

if

$$-\frac{1}{2}\pi \leqq \Re(x) \leqq +\frac{3}{2}\pi, \tag{43b}$$

$$\mathscr{I}(x) \ll 0. \tag{43c}$$

This defining range gives to (40) and (41) a common domain of existence and also to (42) and (43). In order to give a common range to three functions we use equation (9). We get from it and (29) that, for positive large y

$$he^{(1)}\left(\frac{\pi}{2} - iy\right) \sim \frac{\exp\,[2k\,\cosh y + \Phi]}{(2k\,\cosh y)^{1/2}}$$

while from (42) and (43)

$$he^{(3)}\left(\frac{\pi}{2} - iy\right) \sim \frac{\exp\,[2k\,\cosh y - i\pi/2]}{(2k\,\cosh y)^{1/2}},$$

$$he^{(4)}\left(\frac{\pi}{2} - iy\right) \sim \frac{\exp\,[-2k\,\cosh y]}{(2k\,\cosh y)^{1/2}}.$$

We know in addition that a universally valid linear relation must exist between these three functions. In the range considered, the first two are asymptotically large, while the third is asymptotically small. Hence, the relation must read

$$he^{(1)}x = i\,e^{\Phi}\,he^{(3)}x + (\text{unknown factor})\cdot he^{(4)}x. \tag{44}$$

Equation (44) is half a connection formula; we shall now see that the other half can be picked up from consideration of Floquet's theorem.

5. The Floquet Function. According to Floquet's theorem, there exists at least one solution of (1) which is multiplied with a constant factor when we apply the translational symmetry operation of equation (1)

$$x \to x + \pi.$$

This constant is usually written in the form $e^{i\pi\beta}$. It is known that $e^{i\pi\beta}$ oscillates back and forth from the unit circle to the real axis, having alternately positive and negative sign in the latter case. Whenever $e^{i\pi\beta}$ is real we shall use the supplementary definition

$$e^{\pi b} = \pm\,e^{i\pi\beta} \tag{45}$$

the sign being chosen so as to make the real quantity $e^{\pi b}$ positive. The lines along which $e^{i\pi\beta} = \pm 1$ have been studied intensively; a reproduction of the published results

is shown in Fig. 10.[4]. Figure 11 shows a reproduction of a map of β published by McLachlan. This type of information is still rather incomplete at this time.

We now define as $je\ x$ or more specifically $je^+\ x$ the solution of equation (1) which obeys

$$je^+(x + \pi) = e^{i\pi\beta} je^+ x. \tag{46}$$

A second, generally independent, solution $je^-\ x$ is then obtained through

$$je^- x = je^+(-x) \tag{47}$$

which yields the identity

$$je^-(x + \pi) = e^{-i\pi\beta} je^- x. \tag{48}$$

The normalization constant will be left for later disposal. A very important property of $je\ x$ is that whenever $ce\ x$ equals $ce_n x$, je also equals $ce_n x$, and similarly for $se_n x$.

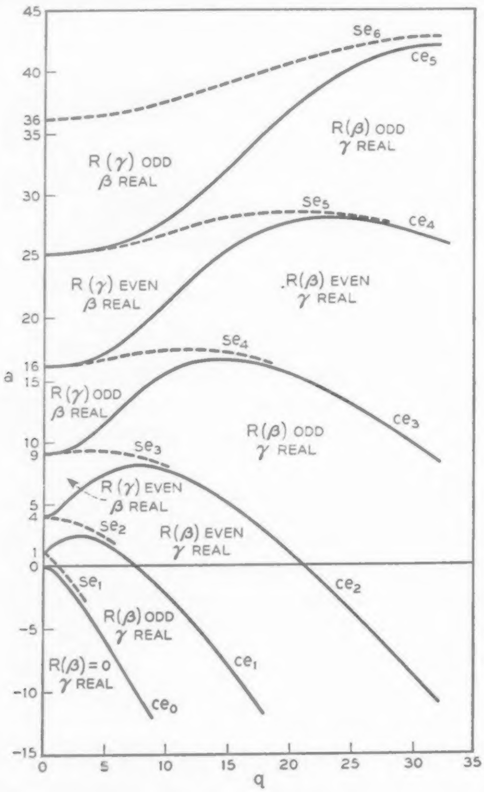

FIG. 10 "BUTTERFLY DIAGRAM" FOR THE MATHIEU FUNCTIONS IN THE a-q PLANE. THE LINES SHOW THE COMBINATION OF VALUES YIELDING PERIODIC SOLUTIONS. INFORMATION CONCERNING THE TWO PHASE ANGLES β AND γ IS ADDED.

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[4]See for instance S. Goldstein, Trans. Camb. Phil. Soc. **23**, 303 (1927); E. L. Ince. Proc. Roy. Soc. Edinburgh **46**, 20 (1925); **46**, 316 (1926); **47**, 294 (1927). Also the textbook of McLachlan. *loc. cit.*

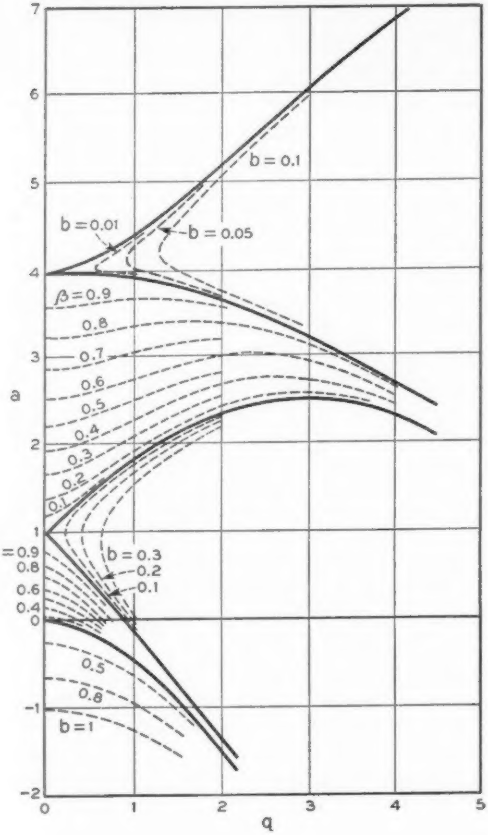

FIG. 11 DIAGRAM SHOWING CURVES OF CONSTANT β IN THE a-q PLANE (McLACHLAN). INFORMATION OF THIS TYPE IS MUCH MORE RESTRICTED THAN THE TYPE SHOWN IN FIG. 10

The asymptotic behavior of the Floquet function was first obtained by Dougall.[5] We shall derive it more quickly by our integral equation method developed in the last section. The pair of equations to be established is

$$B^+ \, je^+ \, x = \int_{u_2 - 2\pi + i\infty}^{u_2 + i\infty} \exp\,[-2ik \cos x \cos u] je^- \, u \, du, \tag{49a}$$

$$B^- \, je^- \, x = \int_{u_2 - 2\pi + i\infty}^{u_2 + i\infty} \exp\,[-2ik \cos x \cos u] je^+ \, u \, du. \tag{49b}$$

u_2 is defined by (38c). The resultant path is shown in Fig. 12 for the case $\Re(x) = 0$.[6]

[5] J. Dougall, Proc. Edinburgh Math. Soc., **34**, 4 (1916); **41**, 26 (1923); **44**, 57 (1926).

[6] The path is identical with the one for the generalized Bessel integral for Bessel functions. The integral permits thus derivation of the Bessel expansion by substituting the Fourier expansion under the integral sign. These relationships suggested to the author the symbol *je* for that type of function.

It is obvious from the discussion preceding equation (38) that as long as $\mathcal{I}(x) > 0$ the two integrals (49) exist and define two solutions of Mathieu's equation. To complete the proof we have to show only that they also obey Floquet's theorem. This is seen as follows. Both terminals of the path lie on the positive imaginary side of the real axis; therefore if $\mathcal{R}(x)$ increases from an initial value, 0 say, the valleys in which the path terminates move in the sense contrary to x, in accordance with equation (36). When x has been increased by 2π the path has been shifted without distortion by an amount -2π. This shift leaves the kernel $\mathcal{K}(x, u)$ of the integral equation invariant, but multiplies the factor $je^- \, u$ in the integrand of (48a) with $e^{2\pi i\beta}$; hence the integral has been multiplied with this same factor and thus obeys Floquet's theorem with the factor $e^{\pi i\beta}$. The same procedure establishes (49b). B^+ and B^- are constants which will be discussed below.

The asymptotic expansion of $je^+ \, x$ and $je^- \, x$ is obtained from (48) by applying the saddle point method discussed in the previous section. The two saddle points lie at $-\pi$ and 0, as shown on Fig. 12. Each saddle point furnishes one of the exponentials

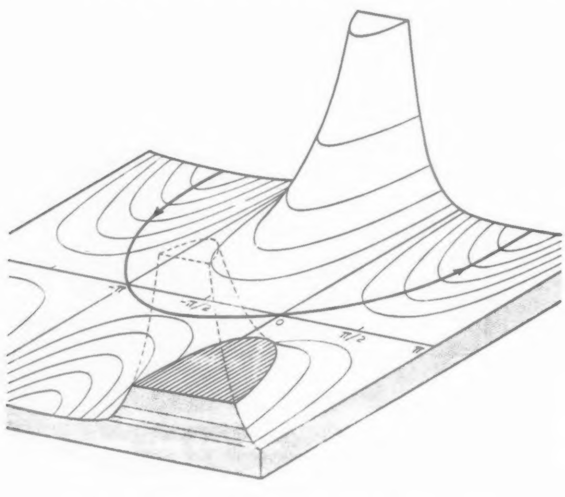

FIG. 12 PATH OF INTEGRATION FOR THE INTEGRALS (49) WHEN X LIES ON THE POSITIVE IMAGINARY AXIS

(33); they can be combined into a single term because we obtain the relative magnitude of the terms from (46) or (48). The resultant expressions are thus found to be

$$B^+ \, je^+ \, x \sim 2 \, je \, 0 \, \exp\left(\tfrac{1}{2} i\pi\beta\right) \left(\frac{\pi}{k \cos x}\right)^{1/2} \cos\left(2k \cos x - \frac{\pi}{4} + \frac{1}{2}\pi\beta\right), \tag{50a}$$

$$B^- \, je^- \, x \sim 2 \, je \, 0 \, \exp\left(-\tfrac{1}{2} i\pi\beta\right) \left(\frac{\pi}{k \cos x}\right)^{1/2} \cos\left(2k \cos x - \frac{\pi}{4} - \frac{1}{2}\pi\beta\right), \tag{50b}$$

with the restriction

$$\mathcal{I}(x) \gg 0. \tag{50c}$$

In the real direction the formulas (50) are also limited. The domain of validity is obtained by the methods used to establish (40c). We find

$$-\frac{\pi}{2} \leqq \mathfrak{R}(x) \leqq +\frac{\pi}{2}. \tag{50d}$$

We have, however, in this case the exceptionally favorable situation that the formulas (46) and (48) just supplement (50d) so as to furnish the asymptotic expansion of *je* x for all values of $\mathfrak{R}(x)$. One simple way to express this is by saying that under the condition (50c) alone $je^{+}\,x$ is asymptotically equal to $J_{-\beta}(ke^{-ix})$, and $je^{-}\,x$ equals $J_{+\beta}(ke^{-ix})$.

The equations (50) would yield the connection formulas between the functions *je* and *he* were it not for the two undetermined constants B^{+} and B^{-}. This indeterminacy can be partially removed by inspection, as follows.

(a) As the B's in (50) are factors in the asymptotic expansion of the *same* function in two different regions of the complex plane, their ratio can never be zero or infinite.

(b) When $je\,x \propto se_n x$ both B's are zero; this follows from symmetry considerations on (49). The formulas (50) do not lose their meaning however, because *je* 0 also vanishes. In these equations symmetry demands that the ratio B^{+}/B^{-} approach -1 as *je* approaches se_n.

(c) When $je\,x \propto ce_n x$, the B's in equation (50) cannot vanish, because *ce* 0 does not vanish. Symmetry or the equations (49) demands that the B's be equal. This means $B^{+}/B^{-} = +1$.

(d) Suppose now we are in a region where $e^{i\pi\beta}$ is real. We then enter into equation (1) with the substitution suggested by (45)

$$me\,x \quad e^{bx}p(x).$$

The resultant equation in $p(x)$ is real; its periodic solution must be real because otherwise there would be two linearly independent periodic solutions. Hence *je* x is a real function of x, and for real *je* 0, the two expressions (50) must be conjugate complex. This gives

$$\frac{B^{+}}{B^{-}} = e^{\pi(b+i\gamma)} \tag{51a}$$

where b is given by (45) and γ is some real number which is integer at the two limiting lines of the region and increases (or decreases) by 1 as we proceed from the line $ce_n\,x$ to $se_n\,x$.

(e) Now let $e^{i\pi\beta}$ be on the unit circle. We then enter into (3) with the substitution

$$me\,x = e^{\beta z}p(z).$$

The resultant equation in z is real. If there is to be only one periodic solution $p(z)$ it must be real along the z direction, and hence the asymptotic expansions (50) must be real. It follows that

$$\frac{B^{+}}{B^{-}} = \pm e^{\pi(i\beta+c)} \tag{51b}$$

where c is some real number which vanishes for integer β. The undetermined sign is fixed in each of the separate regions of real β being $+$ between ce_{2n} and se_{2n+1} and $-$ between ce_{2n+1} and se_{2n}.

(f) We can sum up this information by writing

$$\frac{B^+}{B^-} = e^{i\pi(\beta+\gamma)} \tag{52}$$

where β and γ are functions of a and q which are fixed up to an even integer and the sign. The value of β and γ at the boundary lines of Fig. 10 is shown in Table I. These lines divide up the

Table I Values of β and γ for the Mathieu functions of the first kind

Funtion	β	γ
ce_{2nx}	even	even
ce_{2n+1x}	odd	odd
se_{2nx}	even	odd
se_{2n+1}	odd	even

a-q-plane into "wings" and "gaps". Between these two there is a reciprocal behavior of β and γ. In the wings, β is real and changes by 1; γ does the same thing in the gaps. Inversely, the real part of β is fixed in the gaps, and it has an imaginary part ib which varies; this behavior is duplicated by γ in the wings, where it has a variable imaginary part ic introduced by (51b). This variation is exhibited in Fig. 10.

6. The Connection Formulas. We start out by writing a *formal* connection formula with the help of the parameters β and γ of the last section. We dispose of the normalization factor by setting in accordance with (52)

$$je\, 0 = \frac{B^+ \exp\left[-\frac{1}{2}i\pi(\beta+\gamma)\right]}{2(2\pi)^{1/2}} = \frac{B^- \exp\left[\frac{1}{2}i\pi(\beta+\gamma)\right]}{2(2\pi)^{1/2}}. \tag{53}$$

This reduces (50) to

$$je^+x \sim \frac{\exp\left(-\frac{1}{2}i\pi\gamma\right)}{(2k\cos x)^{1/2}} \cos\left(2k\cos x - \frac{\pi}{4} + \frac{1}{2}\pi\beta\right), \tag{54a}$$

$$je^-x \sim \frac{\exp\left(+\frac{1}{2}i\pi\gamma\right)}{(2k\cos x)^{1/2}} \cos\left(2k\cos x - \frac{\pi}{4} - \frac{1}{2}\pi\beta\right). \tag{54b}$$

The domain of validity is given by (50); it is contained in the larger domains (40) and (41). We may therefore write down the connection formulas

$$je^+x = \tfrac{1}{2}i \exp\left(-\tfrac{1}{2}i\pi\gamma\right)\left[\exp\left(-\tfrac{1}{2}i\pi\beta\right)he^{(1)}x - \exp\left(+\tfrac{1}{2}i\pi\beta\right)he^{(2)}x\right], \tag{55}$$

$$je^-x = \tfrac{1}{2}i \exp\left(+\tfrac{1}{2}i\pi\gamma\right)\left[\exp\left(+\tfrac{1}{2}i\pi\beta\right)he^{(1)}x - \exp\left(-\tfrac{1}{2}i\pi\beta\right)he^{(2)}x\right]. \tag{56}$$

Further, reversing the sign of x with (29), (30), (31), (32) and (47)

$$je^+x = \tfrac{1}{2}i \exp\left(\tfrac{1}{2}i\pi\gamma\right)\left[\exp\left(\tfrac{1}{2}i\pi\beta\right)he^{(3)}x - \exp\left(-\tfrac{1}{2}i\pi\beta\right)he^{(4)}x\right], \tag{57}$$

$$je^-x = \tfrac{1}{2}i \exp\left(-\tfrac{1}{2}i\pi\gamma\right)\left[\exp\left(-\tfrac{1}{2}i\pi\beta\right)he^{(3)}x - \exp\left(\tfrac{1}{2}i\pi\beta\right)he^{(4)}x\right]. \tag{58}$$

Solving (55) and (56) for $he^{(1)}x$ we get

$$he^{(1)}x = \operatorname{cosec}\pi\beta\left[\exp\left\{\tfrac{1}{2}i\pi(\gamma-\beta)\right\}je^+x - \exp\left\{-\tfrac{1}{2}i\pi(\gamma-\beta)\right\}je^-x\right]. \tag{59}$$

Eliminating $je^+ x$ and $je^- x$ from (57), (58) and (59) we get a relation between $he^{(1)} x$, $he^{(3)} x$ and $he^{(4)} x$ which must be identical with the previously derived incomplete equation (44). Comparing coefficients, we find thus

$$i\frac{\sin \pi\gamma}{\sin \pi\beta} = e^{\Phi}, \tag{60}$$

and the completed connection formula (44)

$$he^{(1)}x = i\, e^{\Phi} he^{(3)} x - (i\, e^{\Phi} \cos \pi\beta + \cos \pi\gamma) he^{(4)} x. \tag{61}$$

With equation (60), the formal connection formulas (55)-(59) become actual ones, with coefficients expressible in terms of β and Φ, which, in turn, are known functions of a and q. By the use of the subsidiary definitions (45) and (51) we can show up (60) as an equation between real quantities. Inside the wings of Fig. 10 we get

$$\sinh \pi c = e^{\Phi} \mid \sin \pi\beta \mid, \tag{62a}$$

and in the gaps

$$\sinh \pi b = e^{-\Phi} \mid \sin \pi\gamma \mid. \tag{62b}$$

Actually (60) contains a little more than (62); in the wings for instance, it tells us for each one of the two β's on which side of the real axis the corresponding γ is to be found; a similar sign is determined in the gaps.

Equation (61) effectively terminates the search for continuation formulas because it tells us how to continue the simple exponential asymptotic behavior (40) on the other side of the real axis. A large number of other formulas are derivable from the ones obtained, such as

$$je^+x = \tfrac{1}{2} \exp (\tfrac{1}{2}i\pi\beta - \Phi)[\exp (\tfrac{1}{2}i\pi\gamma)he^{(1)}x + \exp (-\tfrac{1}{2}i\pi\gamma)he^{(4)} x], \tag{63}$$

$$he^{(2)}x = -i\, e^{\Phi} he^{(4)}x + (i\, e^{\Phi} \cos \pi\beta - \cos \pi\gamma)he^{(3)} x, \tag{64}$$

and so forth.

We now come to the connection formulas for the Lindemann-Stieltjes functions. This task has two stages of difficulty. By symmetry alone we can write down relations such as

$$ce\, x \propto je^+x + je^-x,$$

$$ce\, x \propto he^{(1)}x + he^{(3)}x,$$

$$se\, x \propto je^+x - je^-x,$$

$$de\, x \propto \exp (-\tfrac{1}{2}i\pi\beta)je^+x + \exp (+\tfrac{1}{2}i\pi\beta)je^{-x}$$

$$de\, x \propto he^{(1)}x + he^{(4)}x,$$

$$te\, x \propto he^{(1)}x - he^{(4)}x.$$

In order to make connection formulas out of these proportions it is necessary to have quantitative information for the points $x = 0$ and $x = \pi/2$. This paper contains no such information for the point $x = 0$. For the point $x = \pi/2$, on the other hand, the

necessary results were obtained incidentally in section 2. We will pursue this only to the zero stage of the Jeffreys approximation in which we get from (13)

$$ke\,0 \approx \frac{\exp\left(\frac{1}{2}\Phi_0\right)}{(a+2k^2)^{1/4}}, \tag{65}$$

$$ke'0 \approx -\exp\left(\tfrac{1}{2}\Phi_0\right)(a+2k^2)^{1/4}. \tag{66}$$

Hence we may write more precisely

$$de\,x = \frac{1}{2\,ke\,0}\left(he^{(1)}x + he^{(4)}x\right), \tag{67}$$

$$te\,x = \frac{i}{2\mid ke'0\mid}\left(he^{(1)}x - he^{(4)}x\right). \tag{68}$$

The connection formulas which exhibit the asymptotic properties of the Lindemann-Stieltjes functions follow from the ones above by application of (61). We find

$$ce\,x \propto (1 + i\,e^{\Phi})he^{(1)}x - (i\,e^{\Phi}\cos\pi\beta + \cos\pi\gamma)he^{(2)}x, \tag{69}$$

$$se\,x \propto (1 - i\,e^{\Phi})he^{(1)}x + (\cos\pi\gamma + i\,e^{\Phi}\cos\pi\beta)he^{(2)}x. \tag{70}$$

From (40) and (41) it is evident that these two functions always are of the form

$$\frac{1}{(2k\cos x)^{1/2}}\cos\left(2k\cos x - \frac{\pi}{4} + \psi\right) \tag{71a}$$

with

$$-e^{2i\psi} = \frac{\text{second coefficient}}{\text{first coefficient}}. \tag{71b}$$

Perusal of (60) shows that the ψ so defined is always real (not alternating as in (54)). Simple phase shifts of 0 or $\pi/2$ result from (71) when

$$\cos\pi\beta = \pm\cos\pi\gamma = \pm 1$$

in the combination circumstances warranted by Table I; the shifts are then identical with the ones in (54). The remaining relation of the pair (69) and (70) gives us then the asymptotic expansion of the Mathieu function of the second kind whose phase shift comes out to be given by

$$\tan\psi = (-)^{\gamma-1}e^{(-)^{\beta-1}\Phi}.$$

The same formula (61) yields for (67) and (68)

$$de\,x = \frac{1}{2\,ke\,0}\left[-i\,e^{\Phi}he^{(2)}x + (1 - \cos\pi\gamma + i\,e^{\Phi}\cos\pi\beta)he^{(1)}x\right], \tag{73}$$

$$te\,x = \frac{1}{2\mid ke'0\mid}\left[i\,e^{\Phi}he^{(2)}x + (1 + \cos\pi\gamma - i\,e^{\Phi}\cos\pi\beta)he^{(1)}x\right]. \tag{74}$$

These equations yield a real phase shift only in the special circumstances warranted by Table I and equation (54).

The special information (65) and (66) regarding the point $x = \pi/2$ also produces new information about the Floquet function at this point. Using (63), we get

$$je\,\frac{\pi}{2} = \exp\left(\frac{1}{2}\,i\pi\beta - \Phi\right)\cdot ke\;0\cdot\cos\frac{1}{2}\,\pi\gamma, \tag{75a}$$

$$je'\,\frac{\pi}{2} = -\exp\left(\frac{1}{2}\,i\pi\beta - \Phi\right)\cdot|\,ke'0\,|\cdot\sin\frac{1}{2}\,\pi\gamma. \tag{75b}$$

From (75), the connection formula between *je*, *de* and *te* is readily derived.

7. Concluding Remarks. This study is based on the notion that the Floquet parameter β is a known function of a and q. That this is partly a convenient fiction is seen from Fig. 11. It is a surprise that the results of this paper do furnish some new information concerning β. When we traverse the gap between wings from one bounding curve to the other, the exponential damping constant b is related to γ by (62b). On such a path γ changes from one integer to the next and $|\sin\gamma|$ passes therefore through its maximum once. We get therefore the relation for such a path

$$\text{Max}\,[e^{\Phi}\sinh\pi b] = 1. \tag{76}$$

This relation was checked from the Figs. 1, 2 and 11 for a stretch where both quantities are known. The result is Fig. 13 which confirms the prediction (76). In general, $e\Phi$ varies sufficiently slowly so that (76) can be used to determine a rough upper limit for b.

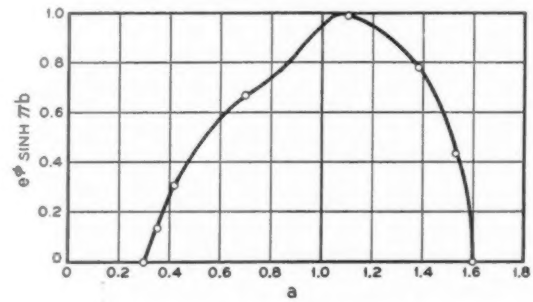

FIG. 13 PLOT OF e^{Φ} SINH πb AGAINST a FOR q = 0.65. ACCORDING TO EQUATION (76) THIS PRODUCT IS BOUNDED BY THE VALUE 1 WHICH IS ACTUALLY ATTAINED AT SOME POINT. THE CRUDENESS OF THE GRAPH REFLECTS THE LIMITED INFORMATION OF FIG. 11.

Notwithstanding this small bit of supplementary information concerning β, there remains the fact that the two natural parameters for the connection formulas are β and γ, and that their dependence on the equation parameters a and q is erratic and not expressible in closed form. It is not likely at this stage that an analytic relation will ever be found connecting β and γ to a and q. For this reason, this paper proceeds instead to find slowly varying and easily determined functions of a and q, from which β and γ can be determined by analytic means. One such parameter is Φ, whose determination was carried out in section 2, and for which the analytic relation to β and γ is equation (60). It is obvious that the accomplishment of the program calls for another such

parameter, to replace β. This parameter has not yet been found. It is interesting to note as a possibility that $\beta + \gamma$ and $\beta - \gamma$ are simpler in their behavior than either one of them alone. In the meantime, the connection formulas of this paper must be used in conjunction with whatever published information is available concerning β.

8. Appendix on Integral Equations. The following integral equations are new, to my knowledge, but proved to be of no use in deriving connection formulas. They may, however, be useful in the hands of others.

Let z, ζ, w be complex numbers whose real parts are x, ξ, u and whose imaginary parts are y, η, v. Then the formulas are

$$B^- je^- z\, je^+ \zeta = je\, 0 \int \exp\,[-2ik \cos z \cos \zeta \cos w - 2k \sin z \sin \zeta \sin w]\, je^+ w\, dw, \tag{77a}$$

$$B^+ je^+ z\, je^- \zeta = je\, 0 \int \exp\,[-2ik \cos z \cos \zeta \cos w - 2k \sin z \sin \zeta \sin w]\, je^- w\, dw. \tag{77b}$$

The formulas are generalizations of (49) to which they reduce for the case $\zeta = 0$. The path and the range are best discussed in two stages. If

$$x - \xi = 0, \tag{77c}$$

then we need

$$y - \eta > 0, \tag{77c$'$}$$

and the path is exactly the one shown in Fig. 12. If

$$x - \xi \text{ arbitrary}, \tag{77d}$$

then we need

$$y - \eta > 0, \qquad \cosh^2(y - \eta) > \frac{4}{3}, \tag{77d$'$}$$

and the abscissa u of the terminal valleys is given by the generalization of (36b)

$$\tan\,(x - \xi) \tanh\,(y - \eta) \tan u = 1. \tag{77d$''$}$$

We set out immediately to prove d and will get c as a special case. We follow earlier proofs quite closely. The formal equivalence of the integral to a product of solutions of the Mathieu equation is found in the literature.[3] Although the integral exists when the exponents cancel, we need a negative real part in the exponent in order to implement the formal steps. This exponent reads

$$\exp = -2ik \cos z \cos \zeta \cos w - 2k \sin z \sin \zeta \sin w \pm 2ik \cos w.$$

The last term arises from the contribution (33) of the Floquet function; the difficult sign is the positive one. By an obvious transformation this becomes

$$\exp = -ike^{-iw} \cos\,(z - \zeta) - ike^{+iw} \cos\,(z + \zeta) + 2ik \cos w.$$

The second term is of no importance because v is to be positive and large. Introducing real and imaginary parts this becomes

$$\exp = -ike^{v}e^{-iu}[\cos\,(x - \xi) \cosh\,(y - \eta) - i \sin\,(x - \xi) \sinh\,(y - \eta)] + ike^{v}e^{-iu},$$

and the exponent's real part is

$$\mathcal{R}(\exp) = -ke^{v}[\cos(x-\xi)\cosh(y-\eta)\sin u + \sin(x-\xi)\sinh(y-\eta)\cos u - \sin u]. \tag{77d''}$$

We now introduce the choice of u indicated by (77d''). This means

$$\sin u = \frac{\cos(x-\xi)\cosh(y-\eta)}{(\cosh^2(y-\eta)+\cos^2(x-\xi)-1)^{1/2}},$$

$$\cos u = \frac{\sin(x-\xi)\sinh(y-\eta)}{(\cosh^2(y-\eta)+\cos^2(x-\xi)-1)^{1/2}}.$$

We thus get

$$\mathcal{R}(\exp) = -ke^{v}\left[(\cosh^2(y-\eta)+\cos^2(x-\xi)-1)^{1/2} - \frac{\cosh(y-\eta)\cos(x-\xi)}{(\cosh^2(y-\eta)+\cos^2(x-\xi)-1)^{1/2}}\right]$$

$$= \frac{-ke^{v}}{(\cosh^2(y-\eta)+\cos^2(x-\xi)-1)^{1/2}}[\cosh^2(y-\eta) + \cos^2(x-\xi) - 1 - \cosh(y-\eta)\cos(x-\xi)].$$

As stated in (77c), the curly bracket is positive when $\cos(x-\xi) = 1$ (and also for $\cos(x-\xi) = 0$); in the general case we form the perfect square

$$[\cos(x-\xi) - \tfrac{1}{2}\cosh(y-\eta)]^2$$

and then pull through on the remainder with (77d'). My surmise is that a more thought-out estimate could prove (77c') all the time. Having proved the character of the function under these restrictions we can determine the particular nature of the left hand side of (77) by first making $\zeta = 0$ and z large and positive, to get the function of z; and then $z = 0$ and ζ large and negative to get the function of ζ. The formulas (77) are thus established. If we reverse (77c') and set instead $y - \eta < 0$ the roles of y and η are reversed because the integral is formally symmetric in z and ζ.

What makes the integral (77) interesting is that one can pass with it from positive to negative imaginary values provided (77c') is maintained. The difficulty in getting asymptotic expressions is in the location of the saddle points; one saddle point of (77a), for instance, lies at $w = \zeta$; this leads to a trivial cancellation and a confirmation of formula (50b). The fact remains, nevertheless, that formula (77) permits us to cross the real axis; this the simpler equations (38) and (49) do not permit us to do.

MIXED BOUNDARY VALUE PROBLEMS IN SOIL MECHANICS[1]

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Summary. The stress-strain law for an ideal soil formulated in a recent paper [1][3] is applied here to obtain the velocity equations referred to the stress characteristic lines in plane strain problems. Simple velocity fields associated with families of straight characteristic lines are then examined, together with discontinuities in the velocity field. The results are applied to obtain the incipient velocity field for the indentation of a semi-infinite mass of material by a flat punch or footing, and to solve the problem of indentation by a lubricated wedge.

1. Introduction. In deriving the solutions of two-dimensional problems in soil mechanics, it is usual to assume that the soil is a plastic material in which slip or yielding occurs when the stresses satisfy the Coulomb formula [2]

$$f = \frac{\sigma_x + \sigma_y}{2} \sin \varphi + \sqrt{\frac{1}{4} (\sigma_x - \sigma_y)^2 + \tau_{xy}^2} - c \cos \varphi = 0, \tag{1}$$

where c is the cohesion and φ is the angle of internal friction of the soil. The stresses also satisfy the equilibrium equations

$$\left.\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} &= 0, \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} &= 0, \end{aligned}\right\} \tag{2}$$

in which the weight of the soil is neglected. The equations (1), (2) are hyperbolic and the two characteristic directions are inclined at an angle $\pi/4 + \varphi/2$ to the direction of the algebraically greater principal stress. In Fig. 1, the lines 1, 2 are the directions of the

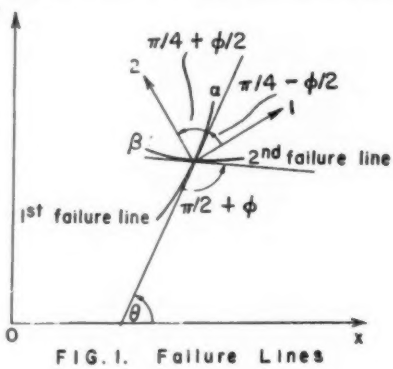

FIG. 1. Failure Lines

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[3]Numbers in square brackets refer to the bibliography at the end of the paper.

principal stresses σ_1, σ_2 ($\sigma_1 < \sigma_2$) at a point and the lines α, β are the characteristic lines passing through the point. We shall call the characteristic line which lies between the 1 and 2 directions the first failure line and denote by θ the angle of inclination of this line to the x-axis.

If we now put

$$p = \frac{(\sigma_2 - \sigma_1)}{2m \sin \varphi} \geq 0, \tag{3}$$

where m is a positive constant which has the dimensions of stress, then, using (1), it can be shown [3] that

$$\left.\begin{aligned} \sigma_x &= -mp[1 + \sin \varphi \sin (2\theta + \varphi)] + c \cot \varphi, \\ \sigma_y &= -mp[1 - \sin \varphi \sin (2\theta + \varphi)] + c \cot \varphi, \\ \tau_{xy} &= mp \sin \varphi \cos (2\theta + \varphi). \end{aligned}\right\} \tag{4}$$

The equations of equilibrium (2) can be replaced by the equations

$$\left.\begin{aligned} &\tfrac{1}{2} \cot \varphi \log p + \theta = \text{const.} \qquad \text{along a first failure line,} \\ &\tfrac{1}{2} \cot \varphi \log p - \theta = \text{const.} \qquad \text{along a second failure line,} \end{aligned}\right\} \tag{5}$$

which were first obtained by Kötter [4].

The three stress components, σ_x, σ_y and τ_{xy}, are determined from the equilibrium equations (2) and the yield condition (1); alternatively, they can be determined by integrating equations (5) along the failure lines. The usual treatment of plane-strain problems assumes that the problems are statically determinate. In general, however, the stress boundary conditions are not sufficient to make the problem statically determinate and a stress-strain law is necessary in order to allow a more complete investigation of the problem. In the following we shall use a stress-strain law which is derived by assuming that the soil is a perfectly plastic body. It should be remarked that this assumption neglects many practical effects (such as the presence of water in the soil). The predictions of the theory must be compared with the actual behavior of soil in order to obtain an indication of the value of the assumption.

2. The velocity field. Drucker and Prager considered a proper generalization of the Coulomb hypothesis (1) and they showed that if the soil is assumed to be a plastic material then, according to the concept of plastic potential [5], the stress-strain law for plane strain corresponding to the yield function (1) is

$$\left.\begin{aligned} \epsilon_x &= \lambda \frac{\partial f}{\partial \sigma_x} = \frac{\lambda}{2} \left\{ \sin \varphi + \frac{(\sigma_x - \sigma_y)/2}{[\frac{1}{4}(\sigma_x - \sigma_y)^2 + \tau_{xy}^2]^{1/2}} \right\}, \\ \epsilon_y &= \lambda \frac{\partial f}{\partial \sigma_y} = \frac{\lambda}{2} \left\{ \sin \varphi - \frac{(\sigma_x - \sigma_y)/2}{[\frac{1}{4}(\sigma_x - \sigma_y)^2 + \tau_{xy}^2]^{1/2}} \right\}, \\ \gamma_{xy} &= \lambda \frac{\partial f}{\partial \tau_{xy}} = \lambda \frac{\tau_{xy}}{[\frac{1}{4}(\sigma_x - \sigma_y)^2 + \tau_{xy}^2]^{1/2}}, \end{aligned}\right\} \tag{6}$$

where ϵ_x, ϵ_y, γ_{xy} are the plastic strain rates and λ is a positive factor of proportionality which may assume different values for different particles. Since we assume that there is

no deformation of the soil until plastic yielding occurs, the plastic strain rate is equal to the total strain rate, and we have

$$\epsilon_x = \frac{\partial u}{\partial x}, \qquad \epsilon_y = \frac{\partial v}{\partial y}, \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x},$$

where u, v are the components of velocity along the x, y-axes. Using equations (4), equations (6) can be written

$$\left.\begin{aligned} \epsilon_x &= \frac{\lambda}{2}\,\{\sin\varphi - \sin(2\theta + \varphi)\}, \\ \epsilon_y &= \frac{\lambda}{2}\,\{\sin\varphi + \sin(2\theta + \varphi)\}, \\ \gamma_{xy} &= \lambda\cos(2\theta + \varphi). \end{aligned}\right\} \tag{7}$$

From the stress-strain relations (6) or (7), the rate of dilation is found to be

$$\epsilon_x + \epsilon_y = \lambda \sin\varphi \geq 0, \tag{8}$$

so that an important feature of the relations is that plastic deformation must be accompanied by an increase in volume if $\varphi \neq 0$.

If we put $\theta = 0$ and $\theta = -(\pi/2 + \varphi)$ in turn in the first equation of equations (7) we find that

$$\left(\frac{\partial u}{\partial x}\right)_{\theta=0} = \left(\frac{\partial u}{\partial x}\right)_{\theta=-(\pi/2+\varphi)} = 0. \tag{9}$$

These equations express the fact that the rate of extension along the failure lines is zero.

The components u, v of the velocity vector are to be determined from equations (7) when the pattern of the failure lines is known for a given plastic stress field. It can easily be shown that the characteristics of the velocities coincide with the characteristics of the stresses and it is more convenient to refer the velocity equations to the characteristic lines. We denote by v_1 and v_2 the *orthogonal projections* of the velocity vector at a point on the directions of the first and second failure lines passing through the point (see Fig. 2). The signs of these velocity projections are chosen so that a counterclock-

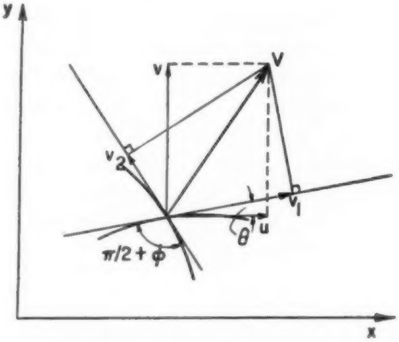

FIG. 2. Velocity Projections

wise rotation of the positive direction along the first failure line through an angle of $\pi/2 + \varphi$ transforms it into the positive direction along the second failure line. The velocity projections v_1, v_2 are related to the cartesian components u, v of the velocity by the equations

$$\left.\begin{aligned} v_1 &= u \cos \theta + v \sin \theta, & v_2 &= -u \sin (\theta + \varphi) + v \cos (\theta + \varphi), \\ u &= \frac{v_1 \cos (\theta + \varphi) - v_2 \sin \theta}{\cos \varphi}, & v &= \frac{v_1 \sin (\theta + \varphi) + v_2 \cos \theta}{\cos \varphi}. \end{aligned}\right\} \tag{10}$$

The substitution of these equations into equations (9), which state that the rate of extension along the failure lines is zero, gives the equations of the velocity field referred to the characteristic lines,

$$\left.\begin{aligned} dv_1 - (v_1 \tan \varphi + v_2 \sec \varphi)\, d\theta &= 0 \qquad \text{along a first failure line,} \\ dv_2 + (v_1 \sec \varphi + v_2 \tan \varphi)\, d\theta &= 0 \qquad \text{along a second failure line.} \end{aligned}\right\} \tag{11}$$

These equations can also be obtained as follows. In Fig. 3, A and B represent two neigh-

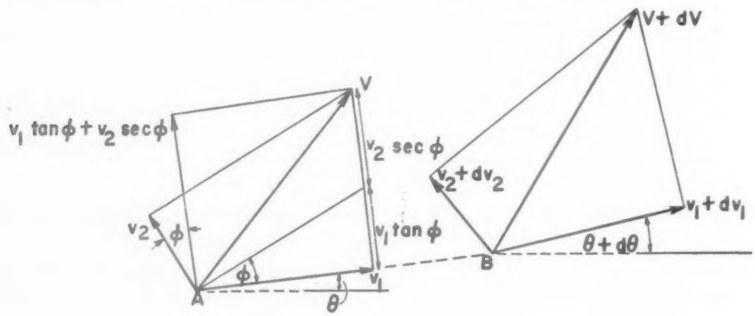

FIG. 3. Inextensibility of failure lines.

boring points on a first failure line separated by an infinitesimal distance ds_1. The end points of the element AB move with the velocities indicated in the figure. The extension of the element may be considered to be caused by the following two circumstances: (i) the velocity along the element at A is smaller than the corresponding velocity at B by an amount dv_1, and (ii) the normal component of the velocity at A and the normal component of the velocity at B are inclined at an angle $d\theta$. The corresponding rates of extension are dv_1/ds_1 and $-(v_1 \tan \varphi + v_2 \sec \varphi)\, d\theta/ds_1$, since the normal component of velocity at A is $v_1 \tan \varphi + v_2 \sec \varphi$. The condition that the total rate of extension along a first failure line must be zero is therefore

$$dv_1 - (v_1 \tan \varphi + v_2 \sec \varphi)\, d\theta = 0$$

along a first failure line. This is the first of equations (11) and the second equation can be obtained in a similar manner.

Equations (11), together with the velocity boundary conditions and the condition that the dilatation must be positive, suffice to determine the velocity field when the failure lines are known.

3. Families of straight failure lines. The simplest pattern of failure lines consists of two families of straight lines intersecting at an angle $\pi/2 + \varphi$. As can be seen from equations (5), this pattern corresponds to a region of constant stress and it is usually called an active Rankine zone or a passive Rankine zone in the literature of soil mechanics.

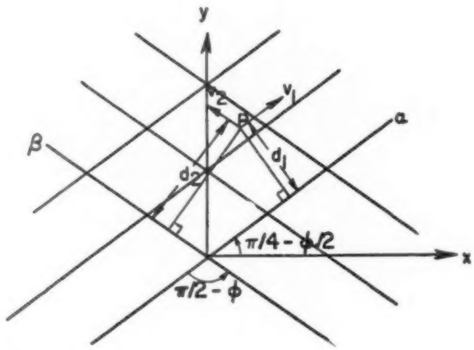

FIG. 4. Region of Constant Stress.

In Fig. 4, the x and y axes are taken along the directions of the minimum and maximum principal stresses respectively. Since θ is constant along the failure lines, equations (11) show that v_1 is constant along a first failure line and v_2 is constant along a second failure line. If we denote by d_1 and d_2 the distances of a current point from one of the first failure lines and one of the second failure lines, say the failure lines which pass through the origin, then we can write

$$v_1 = f(d_1), \qquad v_2 = g(d_2),$$

where

$$d_1 = -x \sin\left(\frac{\pi}{4} - \frac{\varphi}{2}\right) + y \cos\left(\frac{\pi}{4} - \frac{\varphi}{2}\right),$$

$$d_2 = x \sin\left(\frac{\pi}{4} - \frac{\varphi}{2}\right) + y \cos\left(\frac{\pi}{4} - \frac{\varphi}{2}\right),$$

and where the functions f and g are such that the dilatation is positive everywhere in the region. The cartesian components of the velocity vector are found from (10) to be

$$u = \left\{f(d_1) \cos\left(\frac{\pi}{4} + \frac{\varphi}{2}\right) - g(d_2) \sin\left(\frac{\pi}{4} - \frac{\varphi}{2}\right)\right\} \Big/ \cos \varphi,$$

$$v = \left\{f(d_1) \sin\left(\frac{\pi}{4} + \frac{\varphi}{2}\right) + g(d_2) \cos\left(\frac{\pi}{4} - \frac{\varphi}{2}\right)\right\} \Big/ \cos \varphi.$$

When one of the families of failure lines consists of concurrent straight lines, the other family is a system of logarithmic spirals which have the point of intersection as centre. This stress distribution is usually called a zone of radial shear. In Fig. 5 we have

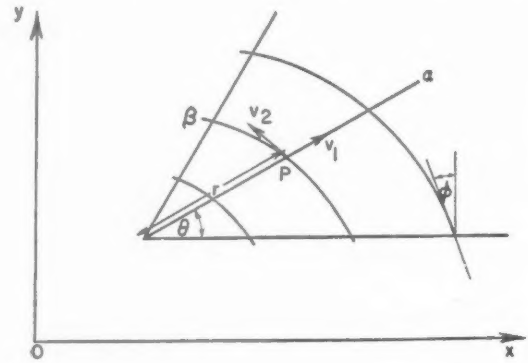

FIG. 5. Zone of Radial Shear.

taken the first failure lines to be the family of straight lines and we denote by r the distance from the centre of the spirals. Since θ is constant along the first failure lines, v_1 is constant along these lines and we have

$$v_1 = f(\theta).$$

Substituting for v_1 in the second of equations (11) we obtain

$$v_2 e^{\theta \tan \varphi} = -\sec \varphi \int_{\theta_0}^{\theta} f(\theta) e^{\theta \tan \varphi} \, d\theta + A$$

along a second failure line. In this equation A is a constant of integration which is constant along each second failure line but which may take different values on different second failure lines. Since the spirals are given by the equation

$$re^{\theta \tan \varphi} = \text{constant},$$

the function A can be written in the form

$$A = g(re^{\theta \tan \varphi}).$$

The functions f and g must be such that the dilatation is positive everywhere in the region. When $f(\theta)$ is zero, i.e., when v_1 is zero, we obtain

$$v_2 = Ae^{-\theta \tan \varphi},$$

so that v_2 varies exponentially along each second failure line.

If the second failure lines had been taken to be the family of concurrent straight lines then we would have obtained

$$v_2 = f(\theta), \qquad v_1 e^{-\theta \tan \varphi} = \sec \varphi \int_{\theta_0}^{\theta} f(\theta) e^{-\theta \tan \varphi} \, d\theta + A,$$

where A is constant along each first failure line. As before, $f(\theta)$ and A must be such that the dilatation is everywhere positive.

4. Discontinuities in the velocity field. It can be shown, as in the theory for a perfectly plastic material, that a line separating a region of plastic flow from a region which remains at rest must be a failure line. This follows from equations (11) because, if the line were not a failure line, a region at rest on one side of the line would imply a certain region at rest on the other side of the line. This still applies if the velocity field is discontinuous across the line separating the two regions. A line of discontinuity in the velocity field must be regarded as a thin layer producing a continuous transition from one velocity field to another. Since a discontinuity in the tangential velocity must be accompanied by a separation or discontinuity in the normal velocity, the transitional layer must have appreciable thickness for a soil while there is no need for such a layer in a Prandtl-Reuss material (for which $\varphi = 0$).

Let P be a point on the median line of such a transitional layer and take the x, y axes along the tangential and normal directions at P. Since, at P, $\partial u/\partial x$ and $\partial v/\partial x$ must be negligible compared with $\partial u/\partial y$, the strain rate ϵ_x must be very small compared with the strain rate γ_{xy}. The stress-strain relations (7) show that this can only be so if we have $\theta = 0$ or π, or $\theta = \pi/2 - \varphi$ or $3\pi/2 - \varphi$, that is, if the line of discontinuity is a first or second failure line respectively. Also from (7), for these values of θ, we have

$$\left.\begin{aligned} \epsilon_y &= \frac{\partial v}{\partial y} = \lambda \sin \varphi, \\ \gamma_{xy} &\doteqdot \frac{\partial u}{\partial y} = \pm\lambda \cos \varphi, \end{aligned}\right\} \tag{12}$$

the positive or negative sign being taken according as the line of discontinuity is a first or second failure line. Equations (12) show that the change in velocity across the line is inclined at angle φ to the line of discontinuity. Further, if we denote by u_1 and u_2 the values of the velocity component u on the sides of the line of smaller and larger values of y, then equations (12) show that $u_2 > u_1$ or $u_1 > u_2$ according as the line of discontinuity is a first or second failure line respectively.

The straight line and the logarithmic spiral of angle φ are the only lines of discontinuity which permit rigid motions, translation and rotation respectively, of the regions separated by the line. These two types of "sliding" discontinuity have been used by Drucker and Prager to obtain upper bounds for the critical height of a vertical bank of soil.

5. Indentation by flat punch. In this section the theory developed above is applied to the indentation of a semi-infinite mass of soil by a flat rigid punch or footing (under conditions of plane strain). A possible plastic stress distribution was determined by Prandtl [6] and we shall consider this stress distribution together with an alternative solution. The two stress distributions, which give the same value for the bearing capacity of the soil, are illustrated in Figs. 6 and 7. They correspond to the two solutions proposed by Prandtl and by Hill [7] for the same problem in a perfectly plastic material (for which $\varphi = 0$). It seems probable that Prandtl's solution is more nearly correct when the punch is sufficiently rough, while the other solution will apply when the surface of the punch is smooth.

Referring to Figs. 6 and 7, plastic regions begin to form at A and B as soon as the load is applied to the punch, but no indentation is possible until the plastic region extends all the way from A to B. We consider only the incipient plastic flow so that the boundary conditions are satisfied at the underformed surface. The problem of determining the stresses and velocities after the punch has penetrated a finite distance is of greater difficulty and would require a study of the successive phases of the plastic flow.

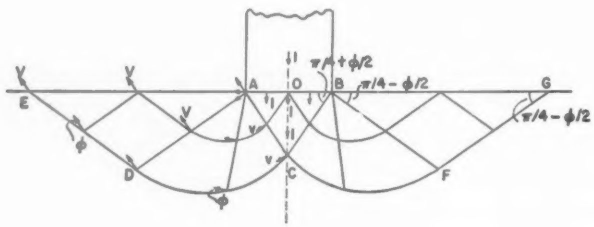

FIG. 6. Prandtl Solution.

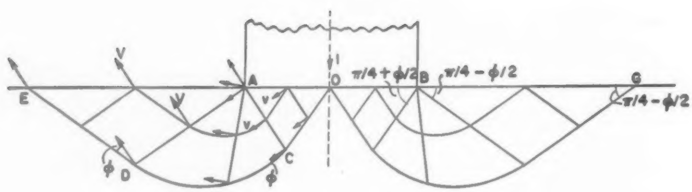

FIG. 7. Alternative Solution

Consider first the solution represented by Fig. 7, and since it is symmetrical about the axis of the punch, we need discuss only the left-hand plastic region. The regions AOC, ADE are regions of constant stress (active and passive Rankine zones respectively), while the region ACD is a zone of radial shear. We take the downward velocity of the punch as the unit of velocity so that along AB the downward component of the velocity of the material must be unity. The material below the second failure line $OCDE$ remains at rest so that $OCDE$ is a line of discontinuity. Thus the velocity along this line is everywhere inclined at an angle φ to the line, i.e., along this line $v_1 = 0$. It follows that $v_1 = 0$ throughout the plastic region since v_1 must be constant along each first failure line.

In the region AOC, v_2 is also constant and the region moves as a rigid body in the direction perpendicular to AC. If v is the velocity of the region then the boundary condition along AO gives

$$v = \sec\left(\frac{\pi}{4} + \frac{\varphi}{2}\right),$$

and we also have

$$v_2 = v \cos \varphi = \cos \varphi \sec \left(\frac{\pi}{4} + \frac{\varphi}{2}\right)$$

in this region. In the zone of radial shear ACD, we have

$$v_2 = Ae^{-\theta \tan \varphi},$$

where A is constant along each second failure line. Since v_2 is constant along AC, we see that v_2 is constant along each first failure line of ACD and also

$$v_2 = \cos \varphi \sec \left(\frac{\pi}{4} + \frac{\varphi}{2}\right) e^{(\tan \varphi)\pi/2}$$

along AD. Finally, the region ADE moves as a rigid body with velocity $V = \sec(\pi/4 + \varphi/2)e^{(\tan\varphi)\pi/2}$ in the direction perpendicular to AD. The velocity field is represented by the small arrows in Fig. 7.

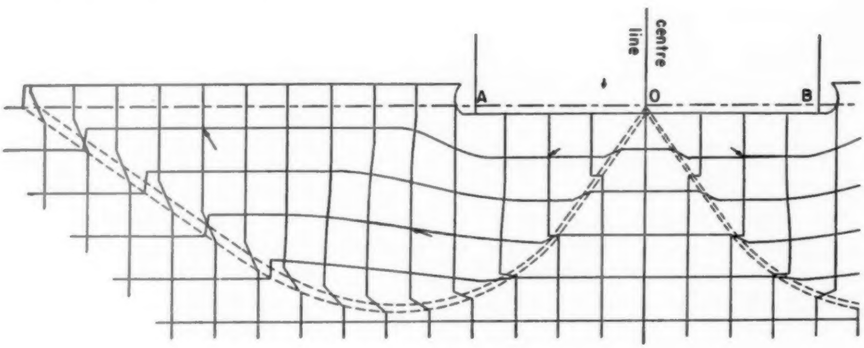

FIG. 8 Resulting deformation of square grid if incipient velocity field was maintained for a short time, according to alternative solution

Figure 8 shows the distortion of a square grid which would result if the initial velocity field was maintained for a short period of time (where φ is taken to be 20°). In obtaining this diagram a thin transition layer was assumed to exist between the line $OCDE$ of Fig. 7 and the material which remains at rest. The initial position of the layer is indicated by broken lines in the figure.

The initial velocity field for the Prandtl solution, Fig. 6, can be obtained in an analogous manner. In this case the region ABC moves downward as a rigid body and the lines AC, BC in addition to the lines CDE, CFG, are lines of discontinuity. As before, we take the downward velocity of the punch as the unit of velocity. Referring to the left-hand side of Fig. 6, the material below the second failure line CDE remains at rest so that the velocity along this line must be inclined at an angle φ to the line. Hence $v_1 = 0$ along this line and since the first failure lines are straight it follows that $v_1 = 0$ in the region $ACDEA$. The velocity of the material just to the left of the discontinuity line AC is perpendicular to AC and its magnitude must be such that the change in velocity across AC is inclined at an angle φ to AC. By drawing the velocity diagram

shown in Fig. 9, we see that the velocity must have the magnitude

$$v = \frac{1}{2} \sec \left(\frac{\pi}{4} + \frac{\varphi}{2} \right).$$

In the zone of radial shear ACD the velocity increases exponentially to the value

$$V = \frac{1}{2} \sec \left(\frac{\pi}{4} + \frac{\varphi}{2} \right) e^{(\tan \varphi) \pi/2}$$

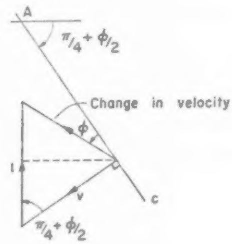

FIG. 9 Velocity diagram for discontinuity line AC in Fig. 6.

on the line AD. The region ADE moves as a rigid body in the direction perpendicular to AD with this velocity V.

The small arrows in Fig. 6 represent the velocity field. The distortion of a square grid which would result if the material moved with this initial velocity field for a short period of time is shown in Fig. 10 (where φ is taken to be 20°). Thin transition layers were assumed to exist between the lines AC, BC and the material in ABC which moves downward with the punch, and a thin transition layer was taken between the line $EDCFG$ and the material at rest. These layers are indicated by the broken lines in Fig. 10.

We notice that the velocity V of the material along AE in the Prandtl solution is

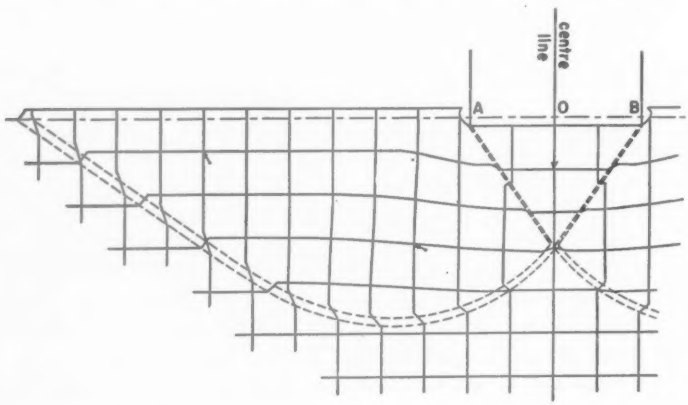

FIG. 10 Resulting deformation of square grid if incipient velocity field was maintained for a short time according to Prandtl solution

exactly half the value of the corresponding velocity in the alternative solution and the velocities are inclined at the same angle, $\pi/4 + \varphi/2$, to the surface AE. For a given width AB of the punch, however, the length of AE in Fig. 6 is twice the length of AE in Fig. 7. It follows that the volume of soil raised above the undisturbed surface after the punch has penetrated a small distance is the same in both solutions.

6. Wedge indentation. We now consider the problem of the indentation of a semi-infinite mass of soil by a smooth, rigid wedge under conditions of plane strain. The solution of the same problem in a perfectly plastic material was obtained by Hill, Lee and Tupper [8], and the solution for the material that we are considering follows this solution closely. Since the configuration is geometrically similar at each stage of the penetration, it is possible to obtain the complete history of the motion without following the deformation step by step.

As the wedge is pressed into the soil, the displaced soil will form a raised lip at each side of the wedge, and the shape of the lips must be determined as part of the solution to the problem. We shall assume that the surfaces of the lips are straight and show that a solution exists which satisfies this assumption. The pattern of failure lines is indicated

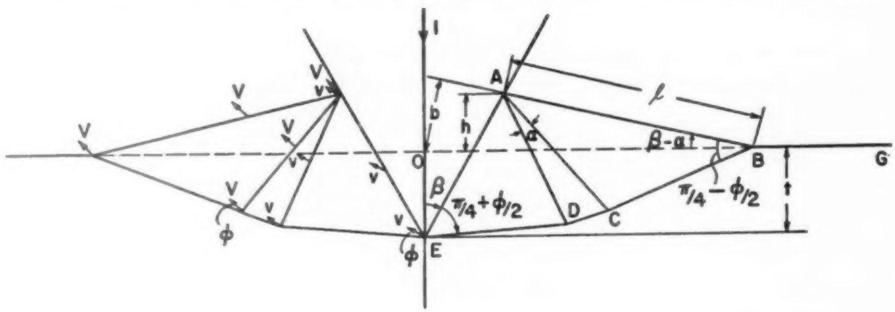

FIG. 11 Indentation by lubricated wedge

in Fig. 11. AE is the right flank of the wedge, which is of angle 2β, BG is the as yet undeformed surface of the soil and AB is the lip. The regions AED, ABC are regions of constant stress while ADC is a zone of radial shear of angle α. We denote by l the length AB of the lip, by b the distance of O from AB, and by h the elevation of A above OB. The depth of penetration of the wedge is denoted by t and if the downward velocity of the wedge is taken as the unit of velocity, we may take t to be the time variable.

The lip AB makes an angle $\beta - \alpha$ with the undisturbed level OB and it is easily shown that we have the following expressions for l, b, h in terms of t, α, β:

$$\left.\begin{aligned} l &= t \Big/ \left\{ e^{-\alpha \tan\varphi} \tan\left(\frac{\pi}{4} - \frac{\varphi}{2}\right) \cos\beta - \sin(\beta - \alpha) \right\}, \\ h &= t \sin(\beta - \alpha) \Big/ \left\{ e^{-\alpha \tan\varphi} \tan\left(\frac{\pi}{4} - \frac{\varphi}{2}\right) \cos\beta - \sin(\beta - \alpha) \right\}, \\ b &= \frac{t \sin(\beta - \alpha) \left\{ e^{-\alpha\tan\varphi} \tan\left(\frac{\pi}{4} - \frac{\varphi}{2}\right) \sin\beta + \cos(\beta - \alpha) \right\}}{\left\{ e^{-\alpha\tan\varphi} \tan\left(\frac{\pi}{4} - \frac{\varphi}{2}\right) \cos\beta - \sin(\beta - \alpha) \right\}}. \end{aligned}\right\} \tag{13}$$

The first failure line $BCDE$ is a line of discontinuity in the velocity field so that the velocity vector along this line must make an angle φ with the line. It follows that $v_2 = 0$ along $BCDE$ and therefore v_2 is zero throughout the plastic region $ABCDE$, since the second failure lines are straight. In the region ADE, v_1 is constant and given by

$$v_1 = v \cos \varphi,$$

where v is the magnitude of the velocity vector in ADE. The boundary condition along AE requires that the velocity of the wedge and that of the soil in contact with it must have the same projection on the normal to AE, and therefore

$$v = \sin \beta \sec \left(\frac{\pi}{4} + \frac{\varphi}{2}\right).$$

In the zone of radial shear the velocity increases exponentially along each first failure line, and along AC the velocity vector has the constant magnitude

$$V = \sin \beta \sec \left(\frac{\pi}{4} + \frac{\varphi}{2}\right) e^{\alpha \tan \varphi}.$$

At a given instant, the region ABC is moving as a rigid body with velocity V in the direction perpendicular to AC. The velocity field in the plastic region is illustrated on the left of Fig. 11.

The projection of the velocity of the lip AB on the normal to AB is

$$V \cos \left(\frac{\pi}{4} - \frac{\varphi}{2}\right) = \sin \beta \tan \left(\frac{\pi}{4} + \frac{\varphi}{2}\right) e^{\alpha \tan \varphi},$$

while the projection of the velocity of the vertex E, which is moving downward with unit velocity, on the normal to AB is $\cos(\beta - \alpha)$. Hence, the distance of E from AB increases at the (constant) rate which is the sum of these two projections. At a time t, i.e., since the beginning of the indentation, the distance of E from AB has therefore reached the value

$$t\left[\sin \beta \tan \left(\frac{\pi}{4} + \frac{\varphi}{2}\right) e^{\alpha \tan \varphi} + \cos (\beta - \alpha)\right]. \tag{14}$$

From Fig. 11 we see that this distance is also equal to

$$b + t \cos (\beta - \alpha), \tag{15}$$

and equating the expressions (14) and (15) gives

$$b = t \sin \beta \tan \left(\frac{\pi}{4} + \frac{\varphi}{2}\right) e^{\alpha \tan \varphi}. \tag{16}$$

The substitution of the expressions (13) into this equation furnishes a relation between the angles α, β, and after some reduction we obtain

$$\cos (2\beta - \alpha) = \frac{\cos \alpha \left[e^{\alpha \tan \varphi} \tan \left(\frac{\pi}{4} + \frac{\varphi}{2}\right) + e^{-\alpha \tan \varphi} \tan \left(\frac{\pi}{4} - \frac{\varphi}{2}\right) \right]}{\left\{ 2 \sin \alpha + e^{\alpha \tan \varphi} \tan \left(\frac{\pi}{4} + \frac{\varphi}{2}\right) + e^{-\alpha \tan \varphi} \tan \left(\frac{\pi}{4} - \frac{\varphi}{2}\right) \right\}}. \tag{17}$$

The variation of the angle α with the angle β, obtained from (17), is shown in Fig. 12 for $\varphi = 20°$.

The pressure P on the flank of the wedge can easily be obtained from the pattern of the failure lines and the boundary condition of zero traction along AB. It is found to be given by

$$P = c \cot \varphi \left\{ e^{2\alpha \tan \varphi} \tan^2 \left(\frac{\pi}{4} + \frac{\varphi}{2} \right) - 1 \right\},$$

where, as before, c is the cohesion of the soil.

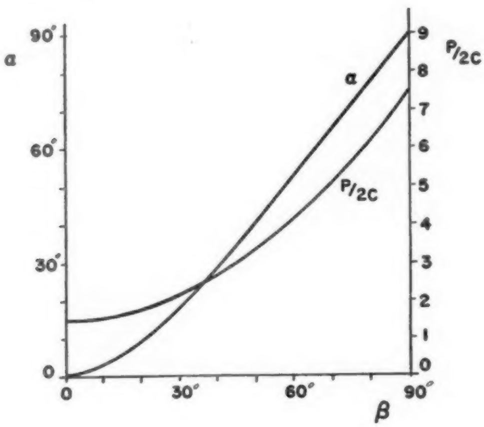

FIG. 12 Variation of P and α with β

The variation of P with the angle β is shown in Fig. 12 for $\varphi = 20°$. Since we have

$$AE = l \tan \left(\frac{\pi}{4} - \frac{\varphi}{2} \right) e^{-\alpha \tan \varphi},$$

the total downward force F necessary to drive the wedge into the soil is given by

$$F = 2Pl \sin \beta \tan \left(\frac{\pi}{4} - \frac{\varphi}{2} \right) e^{-\alpha \tan \varphi}.$$

The determination of the motion of a particular element of the soil appears to be rather complicated since the velocity of the element is influenced by its varying position in space and also by the continual expansion of the velocity field. However, the use of the unit diagram introduced by Hill, Lee and Tupper greatly facilitates the solution. The unit diagram is obtained by transforming the velocity field into a geometrically similar field in which the penetration is always unity. This is effected by the transformation

$$\mathbf{r} = \mathbf{r}^* t,$$

where $\mathbf{r}$ is the actual position vector of the element with reference to O and $\mathbf{r}^*$ is its corresponding position vector in the unit diagram. The method of determining the trajectory of an element is very similar to that used by Hill, Lee and Tupper for the perfectly plastic material and for brevity we omit the details of the solution. The unit diagram,

with three typical trajectories, is given in Fig. 13, and in Fig. 14 we show the deformation of the originally square meshes of a grid, where we have taken $\beta = 30°$, $\varphi = 20°$.

It is also possible to solve the problem when there is a moderate amount of sliding friction between the soil and the wedge. The angle of the wedge must not be too large in order to make it possible to extend the plastic field in a satisfactory manner below the vertex of the wedge. In this case the lines of failure do not meet the flank of the wedge at an angle $\pi/4 + \varphi/2$ but the pressure on the wedge is uniformly distributed. The lip AC is still straight although it is inclined at a smaller angle to the undisturbed surface.

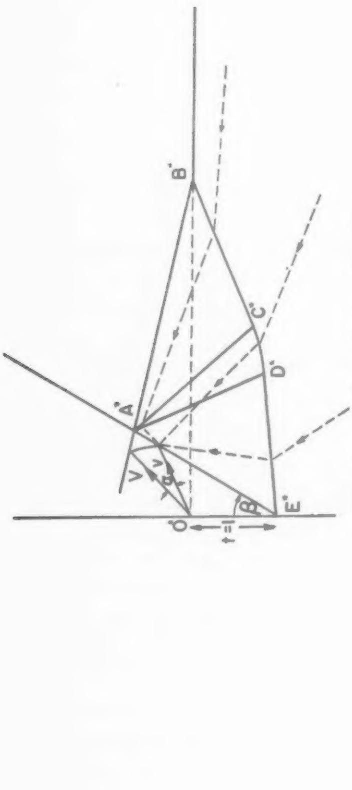

FIG. 13 The Unit Diagram

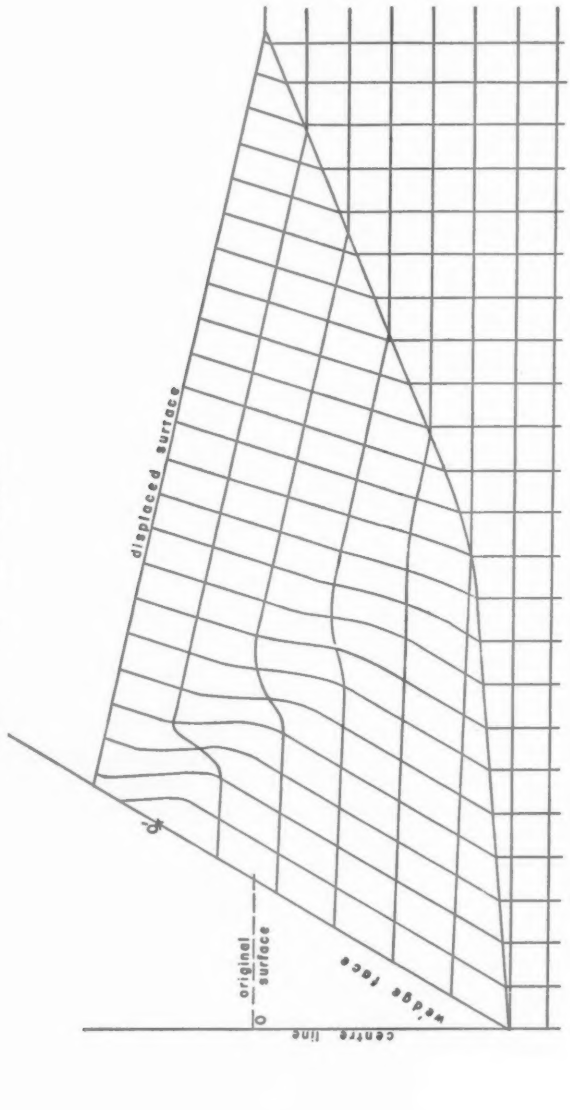

FIG. 14. Deformed grid after indentation by lubricated wedge.

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INFINITE MATRICES ASSOCIATED WITH DIFFRACTION BY AN APERTURE*

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1. Introduction and summary. As an example of their "variational method", LEVINE and SCHWINGER [1] investigated a boundary value problem which arises from the diffraction of a plane scalar (acoustical) wave by a plane screen with a circular aperture. It is equivalent to the problem of finding the field of a freely vibrating circular disk. A full discussion of the physical problems was given by Bouwkamp [2]. Let z, ρ, θ be cylindrical coordinates and let $z = 0$ be the plane occupied by the screen. Let $z = 0$, $0 \leqq \rho < a$ define the aperture (or the vibrating disk). The diffracted field is given by a function u which satisfies $\nabla^2 u + k^2 u = 0$ (with a constant k) everywhere except for $z = 0$ and at infinity satisfies a Sommerfeld radiation condition. For $z = 0$, u must satisfy the "mixed" boundary conditions $u = 0$ for $\rho > a$ and $\partial u/\partial z = v_0$ with a given constant value v_0 for $0 \leqq \rho < a$. These conditions determine u uniquely. For $z = 0$, $0 \leqq \rho < a$, $u = \Phi(\rho)$ becomes a function of ρ only, and if $\Phi(\rho)$ is known or even if only $C_0\Phi(\rho)$ with an undetermined constant factor C_0 is known, u can be determined everywhere; see formulas (A.1), (A.2), (A.3) in [1].

Levine and Schwinger [1] show that the ratio of the energy transmitted through the aperture to the energy incident on the aperture is the imaginary part of the complex transmission coefficient T^*, which is a quotient of two integrals involving $\Phi(\rho)$ quadratically. As a functional of $\Phi(\rho)$, T^* becomes stationary for the correct function Φ which determines u. Levine and Schwinger find approximate values for T^* by expanding first $\Phi(\rho)$ in an infinite series of auxiliary functions (see 3.1 and 3.2) with coefficients D_m. Then T^* becomes a linear form in the D_m (see 3.10), and the unknowns D_m are determined by an inhomogeneous system of infinitely many linear equations with a coefficient matrix L (see 3.4, 3.5). In [1], these equations are solved "section wise", using the first $l = 1, 2, 3, \cdots$ equations to determine the first l unknowns. All quantities D_m, T^*, L are power series in $\beta = ka/2$, and Levine and Schwinger compute the first coefficients of the expansion of T^* in a power series in β which were determined independently by Bouwkamp [2], who used spheroidal wave functions.

It will be shown that the algebraic properties of the matrix L make it possible not only to find approximate values for T^* as in [1] but also to determine $\Phi(\rho)$. This is due to the fact that L factorizes in a product $L^{(0)}S$, where $L^{(0)}$ is the matrix for the static case $k = 0$ and where S can be inverted by solving finite recurrence relations. The details are stated in lemma 1 and theorem 1 of section 3. Lemma 2 gives additional algebraic relations. Problems of convergence and uniqueness are settled in section 5. These depend largely on an investigation of the properties of $L^{(0)}$ which is carried through in section 4. There it is shown that in the limiting cases $k = 0$ and $k = \infty$ the matrices $L^{(0)}$ and $L^{(\infty)}$ of the linear equations also arise from a problem of moments. This also makes it possible to prove that the variational method for the calculation of the transmission

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coefficient will work even for $k = \infty$ where the linear equations for the D_m do not have any solution at all.

2. Notations. The elements of (infinite) matrices are denoted by subscripts n, $m = 0, 1, 2, \cdots$ where n denotes the rows and m denotes the columns. A vector with components x_m is denoted by $\{x_m\}$. We also use the notations

$$(a)_n = \Gamma(a + n)/\Gamma(a) = a(a + 1) \cdots (a + n - 1); \qquad (a)_0 = 1, \tag{2.1}$$

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n, \tag{2.2}$$

where Γ denotes the gamma function and F denotes the hypergeometric series. For results needed here see Whittaker and Watson [3] and Bailey [4].

3. Algebraic properties of the linear equations. Let

$$\Phi(\rho) = -\frac{1}{2} aC_0 \sum_{m=0}^{\infty} x_m(1 - \rho^2/a^2)^{m+1/2} \tag{3.1}$$

be the expansion of the field $\Phi(\rho)$ in the aperture in terms of powers of $1 - \rho^2/a^2$. Here C_0 denotes an undetermined constant and

$$-\tfrac{1}{2}ax_m = D_m \tag{3.2}$$

where the D_m are the unknowns used by Levine and Schwinger [1]. The linear equations for the x_m as obtained from the variational method can be written as follows:

Let $p, q = 0, 1, 2, \cdots$ and let $L^{(2p)}, L^{(2q+3)}$ be infinite matrices with elements $l^{(2p)}_{n,m}$, $l^{(2q+3)}_{n,m}$ defined by

$$l^{(2p)}_{n,m} = (-1)^p \pi^{1/2} A(n, m, p)/B(n, m, p), \tag{3.3}$$

$$l^{(2q+3)}_{n,m} = i(-1)^q \pi^{1/2} A(n, m, q + 3/2)/B(n, m, q + 3/2), \tag{3.4}$$

where, for any values of n, m, t

$$A(n, m, t) = \Gamma(n + 3/2)\Gamma(m + 3/2)\Gamma(n + m + 2t + 1),$$

$$B(n, m, t) = 4\Gamma(t + 1)\Gamma(n + t + 1)\Gamma(m + t + 1)\Gamma(n + m + t + 5/2).$$

Let L be the matrix

$$L = \sum_{p=0}^{\infty} \beta^{2p} L^{(2p)} + \sum_{q=0}^{\infty} \beta^{(2q+3)} L^{(2q+3)}, \tag{3.5}$$

the general element $l_{n,m} = l_{n,m}(\beta)$ of which is a power series in $\beta = \frac{1}{2}ka$. Then

$$\sum_{m=0}^{\infty} l_{n,m} x_m = (n + 3/2)^{-1}. \tag{3.6}$$

Let ξ denote the vector with the components x_m and let $\xi^{(r)}$, $r = 0, 1, \cdots$ be the vector with the components $x^{(r)}_m$ where

$$x_m = \sum_{r=0}^{\infty} \beta^r x^{(r)}_m. \tag{3.7}$$

Let $\eta^{(0)}$ denote the vector with the components $1/(m + 3/2)$. Comparing the coefficients of β^r, $r = 0, 1, \cdots$, on both sides of (3.6) we find

$$L^{(0)}\xi^{(0)} = \eta^{(0)}, \qquad L^{(0)}\xi^{(1)} = 0, \tag{3.8}$$

and, for $r = 2, 3, 4, \cdots$:

$$L^{(0)}\xi^{(r)} + L^{(2)}\xi^{(r-2)} + \cdots + L^{(r)}\xi^{(0)} = 0. \tag{3.9}$$

If

$$T^* = \sum_{m=0}^{\infty} x_m/(m + 3/2), \tag{3.10}$$

the transmission coefficient T becomes

$$T = \beta/2 \operatorname{Im} T^* \tag{3.11}$$

where Im denotes the imaginary part. We shall now show that $L^{(0)}$ is a common left hand factor of all the matrices $L^{(2p)}, L^{(2q+3)}$, such that the right hand factor is a bounded matrix.

Lemma 1. *Let* $p = 1, 2, 3, \cdots$ *and* $q = 0, 1, 2, \cdots$, *and let* $S^{(2p)} = (s_{n,m}^{(2p)})$ and $S^{(2q+3)} = (s_{n,m}^{(2q+3)})$ *be the matrices defined by*

$$\left.\begin{aligned} s_{n,m}^{(2p)} &= 0 \qquad \text{if} \qquad n > p + m \\ s_{n,m}^{(2q+3)} &= 0 \qquad \text{if} \qquad n > q \end{aligned}\right\} \tag{3.12}$$

and otherwise

$$s_{n,m}^{(2p)} = (-1)^n G(n, m, p)/H(n, m, p), \tag{3.13}$$

$$s_{n,m}^{(2q+3)} = i(-1)^q G(n, m, q + 3/2)/H(n, m, q + 3/2), \tag{3.14}$$

where, for any values of n, m, t

$$G(n, m, t) = (-t + 3/2)_n \Gamma(2t - n + m)\Gamma(m + 3/2),$$

$$H(n, m, t) = \Gamma(t + 1)\Gamma(t)\Gamma(t + m - n + 1)\Gamma(t + m + 3/2)(3/2)_n .$$

Then

$$L^{(2p)} = L^{(0)}S^{(2p)}, \qquad L^{(2q+3)} = L^{(0)}S^{(2q+3)}. \tag{3.15}$$

Proof: The element in the n-th row and m-th column of $L^{(0)}S^{(2p)}$ is

$$\frac{\sqrt{\pi}}{4} \frac{(-1)^p}{p!(p-1)!} \frac{\Gamma(n + 3/2)}{n!} \frac{\Gamma(m + 3/2)}{\Gamma(m + p + 3/2)} \sum \tag{3.16}$$

where, because of (2.1) and simple properties of the Gamma function

$$\sum_{n,m} = \sum_{r=0}^{p+m} \frac{(n + r!)}{r!} \frac{\Gamma(r + 3/2)}{\Gamma(n + r + 5/2)} \frac{(-p + 3/2)_r}{(3/2)_r} \frac{\Gamma(2p + m - r)}{\Gamma(p + m - r + 1)} \tag{3.17}$$

$$= \frac{n!\Gamma(2p + m)\Gamma(3/2)}{(p + m)!\Gamma(n + 5/2)} \sum_{r=0}^{p+m} \frac{(n + 1)_r}{r!} \frac{(3/2 - p)_r}{(n + 5/2)_r} \frac{(-p - m)_r}{(1 - m - 2p)_r}. \tag{3.18}$$

The sum in (3.18) can be computed by using Saalschuetz's formula (cf. Bailey [4] for a simple proof) which can be written in the form

$$\sum_{r=0}^{k} \frac{(a)_r(b)_r(-k)_r}{r!(c)_r(1 + a + b - c - k)_r} = \frac{(c - a)_k(c - b)_k}{(c)_k(c - a - b)_k}. \tag{3.19}$$

$(k = 0, 1, 2, \cdots ; c \neq 0, -1 - 2, \cdots \quad -k - 1; 1 + a + b - c \neq 1, 2, \cdots , k)$

Taking $a = n + 1$, $b = -p + 3/2$, $c = 5/2 + n$, $k = p + m$, (3.19) gives for $\sum_{n,m}$ in (3.17)

$$\sum_{n,m} = \frac{n!\Gamma(2p+m)\Gamma(3/2)}{(p+m)!\Gamma(n+5/2)} \frac{(3/2)_{m+p}(n+p+1)_{m+p}}{(n+5/2)_{m+p}(p)_{m+p}}. \tag{3.20}$$

From (3.20) and (3.16) it follows that $L^{(2p)} = L^{(0)}S^{(2p)}$. The proof of $L^{(2p+3)} = L^{(0)}S^{(2p+3)}$ follows by the same method.

The elements of the matrices $S^{(2q+3)}$ are zero except for those in the first q rows. This is not true for the $S^{(2p)}$ but the following lemma shows that $S^{(2p)}$ is a polynomial in $S^{(2)}$ apart from right hand factors which are either the identity or of the type of the $S^{(2q+3)}$.

We have:

Lemma 2. Let $p, t = 1, 2, 3, \cdots$ and let $R^{(t)}$ be the matrix for which the element in the first row and m-th column is

$$\frac{(-1)^{t+1}}{(t-1/2)t!(t-1)!} \frac{\Gamma(m+3/2)\Gamma(2t+m+1)}{\Gamma(m+t+3/2)(t+m+1)!} \tag{3.21}$$

all other elements of $R^{(t)}$ being zero. Then

$$S^{(2)}S^{(2t)} - \frac{t+1}{1-2t} S^{(2t+2)} = R^{(t)}, \tag{3.22}$$

$$S^{(2t+2)} = \sum_{\mu=0}^{t} (-2)^{\mu+1}[(-t+1/2)_{\mu+1}/(-1-t)_{\mu+1}]\{S^{(2)}\}^{\mu}R^{(t-\mu)}, \tag{3.23}$$

where, for $\mu = t$, $R^{(0)}$ denotes $S^{(2)}$. In general,

$$S^{(2p)}S^{(2t)} - \frac{(t+p)!}{p!t!} \frac{\Gamma(3/2)\Gamma(-t-p+3/2)}{\Gamma(-p+3/2)\Gamma(-t+3/2)} S^{(2p+2t)} \tag{3.24}$$

is a matrix in which all elements are zero except those in the first p rows.

The proof of lemma 2 follows again from Saalschuetz's formula. We have now:

Theorem 1. If the equations

$$L^{(0)}\xi^{(0)} = \eta \tag{3.25}$$

have a solution, then all the vectors $\xi^{(m)}$ are determined by $\xi^{(0)}$ and by the relations $\xi^{(1)} = 0$ and the recurrence relations

$$\xi^{(r)} = -S^{(2)}\xi^{(r-2)} - S^{(3)}\xi^{(r-3)} - \cdots - S^{(r)}\xi^{(0)}. \tag{3.26}$$

In the particular case where

$$\eta = \eta^{(0)} = (2/3, 2/5, 2/7, \cdots), \tag{3.27}$$

we have

$$\xi^{(0)} = (8/\pi, 0, 0, 0, \cdots), \tag{3.28}$$

and at most the first $r + 1$ components of $\xi^{(r)}$ are different from zero. $\xi^{(0)}, \cdots, \xi^{(r)}$ are the solutions of the original system (3.6), if we use the first $r + 1$ equations for determining the first $r + 1$ unknowns and thereby neglect all terms involving the higher powers of β from the r-th power onwards. $\xi^{(0)}, \cdots, \xi^{(r)}$ also determine the exact values of the first $r + 1$ coefficients of the expansion of T^* in powers of β.

The proof of theorem 1 follows immediately from lemma 1 and in particular from the fact that the $S^{(2p)}$, $S^{(2q+3)}$ involve many vanishing elements. The uniqueness of the $\xi^{(r)}$, and the existence of the x_m (at least for sufficiently small values of β) will be proved in section 5.

4. Limiting cases for the matrix L. Let

$$P(t) = \Gamma(t + 3/2)/\Gamma(t + 1), \qquad Q(t) = \Gamma(t + 5/2)/\Gamma(t + 1). \tag{4.1}$$

Then Theorem 1 states that the equations

$$\sum_{m=0}^{\infty} l_{n,m}(\beta)x_m = h_n \qquad (n = 0, 1, 2, \cdots) \tag{4.2}$$

can be solved by formal (i.e. not necessarily convergent) power series in β if the equations

$$4L^{(0)}\xi \equiv \left\{\pi^{1/2}P(n) \sum_{m=0}^{\infty} x_m P(m)/Q(n + m)\right\} = \{4h_n\} \tag{4.3}$$

have a solution $x_m = x_m^{(0)}$. We shall investigate (4.3) together with the limiting case $\beta \to \infty$. Levine and Schwinger [1] have shown that then (4.2) tends towards the system of linear equations

$$L^{(\infty)}\xi \equiv \left\{\sum_{m=0}^{\infty} x_m/(n + m + 2)\right\} = \mu\{h_n\}, \qquad (n = 0, 1, 2, \cdots) \tag{4.4}$$

where μ is a constant.

We have to define first the linear space of admissible solutions x_m from the nature of the problem. Since (3.1) is supposed to define the field in the aperture, and since the field cannot have a singularity in the center of the aperture, we must assume that

$$\lim_{\epsilon \to 0} \sum_{m=0} x_m(1 - \epsilon)^m \tag{4.5}$$

exists. Since the original system (3.6) was set up merely in order to define the transmission coefficient, we shall assume that

$$\sum_{m=0}^{\infty} x_m/(m + 3/2) \tag{4.6}$$

converges. This implies, that

$$\sum_{m=0}^{\infty} x_m z^m \tag{4.7}$$

converges for $|z| < 1$ and therefore that the x_m actually define the field in the aperture. Then we prove first:

Lemma 3. *If the vector ξ with the components x_m satisfies* (4.5) *and* (4.6), *then the operators $L^{(0)}$ and $L^{(\infty)}$ are defined for ξ in the sense that the sums in* (4.3), (4.4) *converge for $n = 0, 1, 2, \cdots$*

Proof: Let $Q(t)$ be defined as in (4.1) and let

$$\tau_m = Q(m)/Q(n + m), \qquad \sigma_m = \sum_{r=0}^{m} x_r/(r + 3/2). \tag{4.8}$$

Then the partial sums of the series in (4.3) are

$$\sum_{r=0}^{m} \tau_r x_r/(r + 3/2) = \sum_{r=0}^{m-1} (\tau_r - \tau_{r+1})\sigma_r + \tau_m\sigma_m \tag{4.9}$$

where

$$2\tau_{r+1} - 2\tau_r = 3nP(r+1)/\{Q(n+r)[n+r+5/2]\}. \tag{4.10}$$

Since the $|\sigma_n|$ are bounded and $\sum_r |\tau_r - \tau_{r+1}|$ converges, the sums in (4.3) also converge. The proof for the convergence of the sums in (4.4) is even simpler.

Theorem 2. If the equations $L^{(0)}\xi = \{h_n\}$ *or* $L^{(\infty)}\xi = \{h_n^*\}$ *have a solution* $\xi = \{x_m^{(0)}\}$ *or* $\xi = \{x_m^{(\infty)}\}$ *satisfying* (4.5) *and* (4.6), *then the integral equations*

$$\int_0^1 f(v)(1-v)^{1/2}(1-vz)^{-1}\,dv = 4\pi^{-1/2}\sum_{n=0}^{\infty} z^n h_n n!/(3/2)_n\,, \tag{4.11}$$

$$\int_0^1 f^*(v)v(1-vz)^{-1}\,dv = \sum_{n=0}^{\infty} h_n^* z^n, \tag{4.12}$$

have analytic solutions

$$f(v) = \sum_{m=0}^{\infty} v^m x_m^{(0)}\Gamma(m+3/2)/m!, \qquad f^*(v) = \sum_{m=0}^{\infty} x_m^{(\infty)} v^m. \tag{4.13}$$

The solutions are unique and they also solve the problems of moments

$$\int_0^1 f(v)(1-v)^{1/2}v^n\,dv = 4\pi^{-1/2}h_n n!/(3/2)_n\,, \qquad \int_0^1 f^*(v)v^{n+1}\,dv = h_n^*\,. \tag{4.14}$$

The integrals in (4.11) (4.12) are defined by

$$\int_0^1 = \lim_{\epsilon\to+0}\int_0^{1-\epsilon}. \tag{4.15}$$

Since a formal expansion of the left hand sides of (4.11) and (4.12) leads to the linear equations $L^{(0)}\xi = \{h_n\}$ and $L^{(\infty)}\xi = \{h_n^*\}$, it has only to be shown that, under the assumptions made about the x_m, such an expansion is legitimate. It suffices to prove that

$$\lim_{\epsilon\to 0}\int_0^{1-\epsilon} f(v)(1-v)^{1/2}v^n\,dv = 8\pi^{-1}h_n n!/\Gamma(n+3/2) \tag{4.16}$$

where now $f(v)$ is defined by (4.13) and h_n by $L^{(0)}\xi = \{h_n\}$. Since it follows from the assumption (4.5) about the x_m that $f(v)$ converges absolutely and uniformly for $0 \leqq v \leqq 1-\epsilon$, we may integrate term by term in (4.16). Putting $Y_m = x_m^{(0)}\Gamma(m+3/2)/m!$ this gives (with $v = (1-\epsilon)W$)

$$\begin{aligned}
&\sum_{m=0}^{\infty} Y_m \int_0^{1-\epsilon} v^{n+m}(1-v)^{1/2}\,dv \\
&= \sum_{m=0}^{\infty} Y_m(1-\epsilon)^{n+m+1}\int_0^1 W^{n+m}[1-(1-\epsilon)W]^{1/2}\,dW
\end{aligned} \tag{4.17}$$

$$= \sum_{m=0}^{\infty} Y_m(1-\epsilon)^{n+m+1}(n+m+1)^{-1}F(-1/2, n+m+1, n+m+2; 1-\epsilon) \tag{4.18}$$

$$\begin{aligned}
&= \sum_{m=0}^{\infty} Y_m(1-\epsilon)^{n+m+1}(n+m+1)^{-1}F(-1/2, n+m+1; n+m+2; 1) \\
&\quad + \sum_{m=0}^{\infty} Y_m(1-\epsilon)^{n+m+1}(n+m+1)^{-1}\{F(\cdots; 1-\epsilon) - F(\cdots; 1)\}.
\end{aligned} \tag{4.19}$$

According to Gauss's formula (cf. Whittaker-Watson [3])

$$F(-1/2, n + m + 1; n + m + 2; 1) = (n + m + 1)!\Gamma(3/2)/\Gamma(n + m + 5/2), \quad (4.20)$$

and from Abel's lemma and from lemma 3 it follows that

$$\lim_{\epsilon \to 0} \sum_{m=0}^{\infty} Y_m(1 - \epsilon)^{n+m+1}\Gamma(3/2)(n + m)!/(\Gamma(n + m + 5/2) = 4\pi^{-1/2}h_n n!/(3/2)_n . \quad (4.21)$$

Now we have to show that the second sum in (4.19) tends towards zero as $\epsilon \to 0$. Because of (4.5) it suffices to show that

$$\begin{aligned} c_{m,n}(\epsilon) & \\ = \Gamma(m + 3/2)[m!]^{-1}[n + m + 1]^{-1}\{F(-1/2, n + m + 1; n + m + 2; 1 - \epsilon) & \quad (4.22) \\ - F(-1/2, n + m + 1; n + m + 2; 1)\} & \\ = \Gamma(m + 3/2)(2m!)^{-1} \sum_{k=0}^{\infty} [1 - (1 - \epsilon)^{k+1}] & \\ \cdot (1/2)_k/\{(k + 1)!(n + m + k + 2)\} \to 0 & \quad (4.23) \end{aligned}$$

as $\epsilon \to 0$ uniformly in n, m. We can prove that $| c_{m,n}(\epsilon) | < \epsilon$ by observing that $1 - (1 - \epsilon)^{k+1} \leqq (k + 1)\epsilon$. This and (4.23) gives

$$\begin{aligned} | c_{m,n}(\epsilon) | &\leqq \epsilon\Gamma(m + 3/2)(2m!)^{-1} \sum_{k=0}^{\infty} (1/2)_k(n + m + k + 2)^{-1}\{k!\}^{-1} \\ &= \epsilon\Gamma(m + 3/2)\{2m!(n + m + 2)\}^{-1}F(1/2, n + m + 2; n + m + 3; 1) \\ &= \epsilon\Gamma(1/2)\Gamma(m + 3/2)(n + m + 1)!\{2m!\Gamma(n + m + 5/2)\}^{-1} \quad (4.24) \\ &= \epsilon \frac{\pi^{1/2}}{2} \frac{(m + 1)(m + 2) \cdots (m + n + 1)}{(m + 3/2)(m + 5/2) \cdots (m + n + 3/2)} \leqq \epsilon\pi^{1/2}/2 < \epsilon. \end{aligned}$$

The uniqueness of the solution follows from

Lemma 4: *If* $\sum_{m=0}^{\infty} x_m/(m + 3/2)$ *converges, then for* $0 \leqq v < 1$, $(1 - v)^{3/2} | f(v) |$ *is bounded.* The proof follows from summation by parts with the notation (4.8) and from the remark that

$$\sum_{m=0}^{\infty} \Gamma(m + 5/2) | \sigma_m | v^m/(m + 1)! \leqq C[(1 - v)^{-3/2} - 1]v^{-1}, \quad (4.25)$$

where c does not depend on v.

Now we can show that (4.3) cannot have a null solution. Because then the difference $\phi(v)$ of two solutions of (4.11) would satisfy

$$\int_0^1 \phi(v)(1 - v)^{1/2}v^n \, dv = 0, \qquad n = 0, 1, 2, \cdots , \quad (4.26)$$

and therefore:

$$\int_0^1 \phi(v)(1 - v)^{1/2}(1 - v)v^n \, dv = 0, \qquad n = 0, 1, 2, \cdots \quad (4.27)$$

But $\phi(v)$ $(1 - v)^{3/2}$ would be a function continuous in $0 \leqq v \leqq 1$ according to lemma 4 and therefore (4.27) shows that $\phi(v)(1 - v)^{3/2}$ would be identically zero.

Conclusions from theorem 1. The equivalence of the equations $L^{(0)}\xi = \{h_m\}$ and $L^{(\infty)}\xi = \{h_m^*\}$ to a problem of moments shows that these sets of linear equations are unstable in the following sense: Not only may these equations have no solution at all, but this is certain to happen if we start with a set $\{h_m\}$ of right hand sides for which a solution exists and then change a finite number of the h_m by an amount however small. In this case there does not even exist a continuous function $f(v)$ which satisfies (4.11) or (4.12) with the modified right hand sides.

The integral operators in (4.11), (4.12) are extensions of the linear operators defined by $L^{(0)}$ or $L^{(\infty)}$, since (4.11) or (4.12) may have a continuous solution $f(v)$ which is not analytic. Consequently, a quantity like the transmission coefficient

$$T^* = \int_0^1 f(v)v^{1/2}\,dv = \sum_{m=0}^{\infty} x_m/(m + 3/2) \tag{4.28}$$

can be defined even in cases where the x_m do not exist. An easy example is offered by the equations

$$\sum_{m=0}^{\infty} x_m/(n + m + 2) = \mu/(n + 3/2), \qquad (n = 0, 1, 2, \cdots) \tag{4.29}$$

which were also investigated by Levine and Schwinger. The corresponding integral equation is

$$\int_0^1 f(v)v(1 - vW)^{-1}\,dv = \mu \sum_{n=0}^{\infty} W^n/(n + 3/2) = \mu \int_0^1 v^{1/2}/(1 - vW)^{-1}\,dv \tag{4.30}$$

which gives

$$f(v) = \mu v^{-1/2}, \qquad T^* = \mu. \tag{4.31}$$

In this case no set of x_m satisfying (4.29) can exist. However, it is possible to find sequences of constants $Y_m^{(r)}$ such that

$$\sum_{m=0}^{\infty} Y_m^{(r)}(m + n + 2)^{-1} = \psi_n^{(r)} \tag{4.32}$$

exist and

$$\lim_{r\to\infty} \sum_{n=0}^{\infty} \{\psi_n^{(r)} - \mu/(n + 3/2)\}^2 = 0, \qquad \lim_{r\to\infty} \sum_{m=0}^{\infty} Y_m^{(r)}/(m + 3/2) = \mu. \tag{4.33}$$

For this purpose, we can choose the $Y_m^{(r)}$ from

$$\sum_{m=0}^{\infty} Y_m^{(r)} v^m = \sum_{k=0}^{r} (1 - v)^k (1/2)_k/k! \tag{4.34}$$

The right hand side in (4.34) is a polynomial which approximates $v^{-1/2}$, since it is the $(r + 1)$-th partial sum of $[1 - (1 - v)]^{-1/2}$. Clearly, the $Y_m^{(r)} \to \infty$ as $r \to \infty$.

5. Uniqueness and existence of the solution. Once a vector $\xi^{(0)}$ has been determined such that $L^{(0)}\xi^{(0)} = \eta$, where η is the vector of the right hand sides in the original equations $L\xi = \eta$, we can determine ξ from

$$M\xi = \xi^{(0)} \tag{5.1}$$

where, for all values of β, M is defined by

$$M = \mathcal{I} + \sum_{p=1}^{\infty} \beta^{2p} S^{(2p)} + \sum_{q=0}^{\infty} \beta^{2q+3} S^{(2q+3)} \tag{5.2}$$

Here $\mathcal{I}$ denotes the identity. We shall call a vector ξ bounded if $\sum |\xi_m|^2 < \infty$ and we shall call a matrix M bounded if there exists a constant $U > 0$ such that for all bounded vectors ξ:

$$\xi^* M'^* M \xi \leqq U^2 \sum |\xi_m|^2 \tag{5.3}$$

where M' is the transposed matrix of M and an asterisk denotes the conjugate complex quantity. U is called an upper bound for M. It is well known that, if U_r is an upper bound for $S^{(r)}(r = 1, 2, 3 \cdots)$, the matrix M in (5.2) has a bounded inverse M^{-1} if

$$\sum_{r=2}^{\infty} \beta^r U_r < 1 \tag{5.4}$$

M^{-1} can be obtained from a Neumann series. We can use this in order to prove:

Theorem 3. *Let* L, M, $\eta^{(0)}$, $\xi^{(0)}$ *be defined by* (3.5), (5.1), (3.27), (3.28). *Then* M^{-1} *exists and is bounded for sufficiently small values of* $|\beta| < \beta_0$ *and the equations* $L\xi = \eta^{(0)}$ *have exactly one solution* ξ *which satisfies* (4.5) *and* (4.6), *namely* $\xi = M^{-1}\xi^{(0)}$.

Proof: Let $V^{(r)}$ be matrices such that

$$\left\{\mathcal{I} + \sum_{r=2}^{\infty} \beta^r S^{(r)}\right\}\left\{\mathcal{I} + \sum_{r=0}^{\infty} \beta^r V^{(r)}\right\} = \mathcal{I}. \tag{5.5}$$

It is easily seen that the $V^{(r)}$ can be obtained from the $S^{(r)}$ by recurrence formulas. Let $U^{(r)}$ be upper bounds for the $S^{(r)}$ and assume that there exist constants Ω_r such that

$$\left(1 - \sum_{r=2}^{\infty} \beta^r U_r\right)\left(1 + \sum_{r=2}^{\infty} \beta^r \Omega_r\right) = 1. \tag{5.6}$$

This is true if

$$1 - \sum_{r=2}^{\infty} \beta^r U_r \tag{5.7}$$

is convergent and positive for $0 \leqq \beta < \beta_0$. Then it can be shown that Ω_r is an upper bound for $V^{(r)}$. Since it can also be shown that x_m (the m-th component of $\xi = M^{-1}\xi^{(0)}$) is equal to the m-th component of

$$\left\{\sum_{r=m}^{\infty} \beta^r V^{(r)}\right\}\xi_0 \tag{5.8}$$

it follows that

$$|x_m| \leqq \sum_{r=m}^{\infty} \beta^r \Omega_r . \tag{5.9}$$

From this it can easily be shown that for $|\beta| < \beta_0$ condition (4.5) for the x_m is satisfied. This proves the existence of M^{-1} and of a bounded ξ satisfying (4.5), (condition (4.6)

is always satisfied for bounded ξ) if we can find U_r which are sufficiently small. We have

Lemma 4. The matrices

$$\{S^{(2)}\}^t, \qquad R^{(t)}, \qquad S^{(2t+2)}, \qquad S^{(2q+3)} \tag{5.10}$$

have as upper bounds

$$\pi(\pi^2 - 8)^{1/2}/4, \qquad 2^{1/2}(\pi^2 - 8)^{1/2}/t!, \qquad (2\pi^2 - 16)^{1/2}2^{t+2}(1/2)_t/(t + 1)!,$$
$$2^{q+1}(2\pi^2 - 16)^{1/2}/(q + 1)! \tag{5.11}$$

The proof is elementary but laborious and will be omitted since the upper bounds are not the best possible ones.

In order to prove the uniqueness of the solution $\xi = M^{-1}\xi^{(0)}$ we observe first that $(M - \mathfrak{g})\xi$ is bounded for every ξ merely satisfying (4.5); provided that β is so small that (5.4), with the U_r from Lemma 4, converges. This can be proved by an elementary investigation of the $S^{(r)}$. Now if there is a ξ^* satisfying (4.5) and (4.6) such that $L\xi^* = 0$, we would have $M\xi^* = \xi^* + \zeta$ where ζ is bounded and $L^{(0)}\xi^* + L^{(0)}\zeta = 0$. Now it follows from the equivalence of the operator $L^{(0)}$ to the operator of a moment problem (cf. Theorem 2) that $\xi^* + \zeta = 0$. Therefore ξ^* is bounded, and since M^{-1} is bounded, ξ^* must be zero since $M\xi^* = \xi^* + \zeta = 0$.

No numerical values for the permissible ranges of β are given since it is entirely possible that the inverse M^{-1} exists for all values of β. This seems to be indicated by a result of Sommerfeld and Perron [5] who showed that for the related problem of the freely vibrating disc the real part of a resulting set of linear equations can be solved explicitly and without restrictions.

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A GENERALIZATION OF MODULATION SPECTRA*

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I. Introduction. A general theory of modulation spectra may be developed by the use of Fourier analysis. It may be applied to frequency as well as to amplitude modulation and is particularly valuable in the study of modulation products resulting when nonlinear devices such as rectifiers are used as modulators. In all cases it shows that the modulation products are harmonics of the highest common factor among the carrier and the modulating frequencies. Also, this approach yields some new results and some clarification of concepts.

Of course the methods of Fourier can only be used where there is an integral relationship between the carrier and each modulating frequency so that the modulated wave may be treated as a periodic function. When this is not strictly true Bohr's method for almost periodic functions may be used.

II. Outline of Theory.

1. *Modulation products.* When two or more waves are combined in a nonlinear circuit such as a diode rectifier, a reactance-tube oscillator, or the human ear, new frequencies appear as a result of some characteristic (such as amplitude or frequency) of one wave being modified by another. Mathematically the process may be expressed as

$$e = F(e_1, e_2, \cdots, e_n). \tag{1}$$

The new waves, which may include waves of the same frequency as the original waves, are the modulation products.

The principal ways in which modulation may be achieved for the simple case of two input waves are:

(a) Mixing in a nonlinear circuit whose characteristic is representable by a finite number of terms of a power series.

$$F_1(e_1, e_2) = \sum_{m=0}^{m=k} (ae_1 + be_2)^n. \tag{2}$$

(b) Mixing in a nonlinear circuit consisting of a biased ideal rectifier whose forward characteristic is representable by a finite number of terms of a power series.

$$F_2(e_1, e_2) = 1[ae_1 + be_2 - E] \sum_{m=0}^{m=k} (ae_1 + be_2 - E)^m. \tag{3}$$

(Here 1[ ] is the Heaviside unit function. The summation is zero unless the term in square brackets is positive.)

(c) Amplitude Modulation

$$F_3(e_1, e_2) = \sum_{m=m_1}^{mk} \sum_{n=n_1}^{nh} (a_m + b_m e_1)^m (a_n + b_n e_2)^n. \tag{4}$$

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(d) Angle Modulation

$$F_4(e_1\,, e_2) = K \begin{matrix}\sin\\ \text{or}\\ \cos\end{matrix} \{f_1(e_1\,, e_2) + f_2(e_1\,, e_2)\}. \tag{5}$$

Particular values of the constants in these expressions reduce them to ones which are more familiar in engineering practice. Thus if in (4) $a_n = 0$, $m = n = 1$, one has ordinary amplitude modulation.

$$F_3(e_1\,, e_2) = a_m b_n(1 + ke_1)e_2\,, \tag{6}$$

where $K = b_m a_n$ is the modulation factor.

In (5) if $f_1(e_1\,, e_2) = k_1 \sin^{-1} e_1/ \mid e_1 \mid \equiv k_1\omega t$, and $f_2(e_1\,, e_2) = k_2\, e_2$ then one has ordinary phase modulation, or

$$F_4(e_1\,, e_2) = K \begin{matrix}\sin\\ \text{or}\\ \cos\end{matrix} (k_1\omega t + k_2 e_2). \tag{7}$$

If the frequencies of the waves which are combined in a nonlinear circuit are commensurable (in the language of electrical engineering) or contain a common factor or factors, then the modulation products will have a common period which is given by the highest common factor among the original frequencies. In this case ordinary Fourier analysis may be used to find the spectral components.

It is possible that the frequencies of the combined waves may be incommensurable or have no common factor. In such cases (1) may be treated as an "almost periodic function", the theory of which was first advanced by H. Bohr in 1925 [1]. Such a wave would never repeat itself exactly, but for any small quantity ϵ there is always an approximate period τ at the beginning and end of which the amplitudes of the wave differ by less than ϵ. There are actually infinitely many such periods.

The expansion of the almost periodic function is called a generalized Fourier series whose coefficients are found by a limiting process as follows:

$$a_n = \lim_{T\to\infty} \frac{2}{T} \int_0^T f(t) \cos \lambda_n t\, dt, \tag{8}$$

$$b_n = \lim_{T\to\infty} \frac{2}{T} \int_0^T f(t) \sin \lambda_n t\, dt. \tag{9}$$

It is to be noted that it is no longer necessary to ascertain λ_n beforehand. If one replaces λ_n in (8) by some variable x, the limit will be in general zero. The λ_n's are then the values of x which could render these limits not identically zero.

Therefore, to sum up, we see that the modulation products can always be analyzed into systematic spectral components by finding the Fourier or generalized Fourier coefficients. The result is often more revealing than the conventional trigonometrical expansion used in engineering.

An important theorem on Fourier coefficients known as Parseval's theorem [2] will be of use in the development of energy changes due to modulation. In its simplest form, this theorem states that if a function $f(x)$ has its square summable in $(-\pi, \pi)$ and if its

Fourier coefficients are $a_0/2, a_1, a_2, \cdots b_1, b_2, b_3, \cdots$ then

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \{f(x)\}^2\, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2). \tag{10}$$

This theorem fits our problem because it is well known that the average power or energy of an electric wave is proportional to the sum of the squares of the amplitudes of its Fourier components.

Bohr's "Fundamental Theorem" has proved that the generalized Fourier series for almost periodic functions still satisfies Parseval's formula, providing again a theoretical basis for considering the energy changes in modulation.

III. Application of the Method and New Results. The method will now be applied to the analysis of several typical examples of modulated waves whose conventional sideband expansions are known. The new findings and clarification of concepts will be evident where they occur.

1. *Modulation products with pure period.*

a. *Simple Amplitude-Modulated Wave.* This is usually given in the form

$$e = A(1 + m \cos \omega_a t) \cos \omega_c t, \tag{11}$$

where A is the carrier amplitude,
ω_a is the modulation frequency,
ω_c is the carrier frequency,
m is the degree of modulation.

By a trigonometrical identity (11) can be written as

$$e = A \cos \omega_c t + (mA/2) \cos (\omega_c + \omega_a)t + (mA/2) \cos (\omega_c - \omega_a)t. \tag{12}$$

It can be shown that these sidebands are actually harmonics of a certain fundamental in (11) as follows. Let the highest common factor between ω_c and ω_a be ω_0, such that $\omega_c = n_c\omega_0$, $\omega_a = n_a\omega_0$. Then $T = 2\pi/\omega_0$, and

$$\begin{aligned} a_n &= \frac{4}{T}\int_0^{T/2} A(1 + m \cos n_a\omega_0 t) \cos n_c\omega_0 t \cos n\omega_0 t\, dt, \\ b_n &= 0. \end{aligned} \tag{13}$$

The cosine coefficient in (13) gives

$$\begin{aligned} a_n = \frac{2}{\pi}\int_0^{\pi} A \cos n_c\omega_0 t \cos n\omega_0 t\, d\omega_0 t \\ + \frac{1}{\pi}\int_0^{\pi} mA[\cos (n_c + n_a)\omega_0 t + \cos (n_c - n_a)\omega_0 t] \cos n\omega_0 t\, d\omega_0 t. \end{aligned} \tag{14}$$

Thus a_n is not zero only when $n = n_c$ and when $n = n_c + n_a$ giving

$$a_{n_c} = A, \qquad a_{n_c \pm n_a} = mA/z. \tag{15}$$

Therefore the three terms in (12) are the (n_c)th, the $(n_c + n_a)$th and the $(n_c - n_a)$th harmonics of a wave of period $2\pi/\omega_0$ whose fundamental and other harmonics are zero.

b. *Simple Frequency-Modulated Wave.* Assume again the simplest form with conventional notation,

$$e = A \sin (\omega_c t + m_f \sin \omega_a t). \tag{16}$$

Since this is an odd function, one can safely ignore all the cosine coefficients. Then if ω_0 is again the highest common factor between ω_c and ω_a,

$$b_n = \frac{2}{\pi} \int_0^\pi A \sin (n_c\omega_0 t + m_f \sin n_a\omega_0 t) \sin n\omega_0 t \, d\omega_0 t. \tag{17}$$

If we use the identities,

$$\cos (m_f \sin x) = J_0(m_f) + 2 \sum_{k=1}^{\infty} J_{2k}(m_f) \cos 2kx, \tag{18a}$$

$$\sin (m_f \sin x) = 2 \sum_{k=0}^{\infty} J_{2k+1}(m_f) \sin (2k + 1)x, \tag{18b}$$

then (17) can be expanded and integrated to give

$$b_n = A[J_{s_1}(m_f) - J_{s_2}(m_f)](-1)^{s_1} \tag{19}$$

where $s_1 = (n_c - n)/n_a$ has values which are positive or negative integers including zero and $s_2 = (n_c + n)/n_a$ has values which are positive integers and $J_s(m_f)$ is the Bessel coefficient of the first kind of order s and argument m_f. When s is negative,

$$J_s(m_f) = (-1)^{-s} J_{-s}(m_f). \tag{20}$$

There are special values for n_a for which (19) will actually involve two terms as given but otherwise there will be but one term. This can be seen as follows: n_c and n_a are prime to each other and therefore both are odd numbers or one is odd and the other even. Because $n_c = n_a(s_1 + s_2)/2$ where s_1 and s_2 are integers then for n_a and n_c both odd it is only possible to have $n_a = 1$. If n_a is even and n_c is odd it is only possible to have $n_a = 2$ while if n_a is odd and n_c even it is again only possible to have $n_a = 1$.

In view of this, equation (19) can be written as follows:

If $n_a = 1$, whether n_c is even or odd,

$$b_n = A[J_{n_c-n}(m_f) - J_{n_c+n}(m_f)](-1)^{(n_c-n)/2}. \tag{21}$$

If $n_a = 2$, and n_c is odd,

$$b_n = (A/2)[J_{n_c-n}(m_f) - J_{n_c+n}(m_f)](-1)^{(n_c+n)/2}. \tag{22}$$

If $n_a = 1$ or 2, n_c either even or odd,

$$b_n = (A/n_a)[J_{n_c-n}(m_f)](-1)^{(n_c-n)/n_a}. \tag{23}$$

To see what these coefficients really mean, take $n_c = 26$, $n_a = 7$. If $n = 1$, (i.e. consider the fundamental of the wave), then $(n_c \pm n)/n_a = (26 \pm 1)/7$ is not an integer indicating that the frequency component at ω_0 is zero.

Next let $n = 2$; then $(n_c - n)/n_a = (26 - 2)/7 \neq$ an integer, and $(n_c + n)/n_a = (26 + 2)/7 = 4$ and $(-1)^4 = 1$ indicating that the second harmonic at $2\omega_0$ of magnitude $-AJ_4(m_f)$ exists.

A continuation of this process will show that the fifth harmonic exists and is of magnitude $-AJ_3(m_f)$. The ninth harmonic exists and is of magnitude $AJ_5(m_f)$ and the

twelfth harmonic exists and is of magnitude $AJ_2(m_f)$, and etc. This result is shown in the diagram of Fig. 1.

It is interesting to note that this is the same spectrum as if in the ordinary expansion those side-bands of negative frequencies were reflected at the zero frequency axis with signs reversed. This diagram also shows that in general there will be additional frequency components sandwiched between the ordinary sideband spaces, for example, those between the carrier J_0 and the first sidebands J_1.

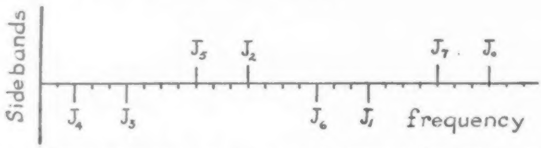

FIG. 1. FM sidebands, $n_c = 26$, $n_a = 7$, (magnitudes not to scale).

Another defect of the ordinary expansion appears in the special case when $n_a = 1$ or when $n_a = 2$ and n_c is an odd number. Thus, if $n_a = 1$, $n_c = 9$, say, then application of (19) will show that the fundamental is of magnitude

$$A[J_8(m_f) - J_{10}(m_f)],$$

and the second harmonic is of magnitude

$$-A[J_7(m_f) - J_{11}(m_f)].$$

And if $n_a = 2$, $n_c = 9$, the application of (19) will show that even harmonics do not exist and the odd harmonics each involve two terms. These are shown in Fig. 2.

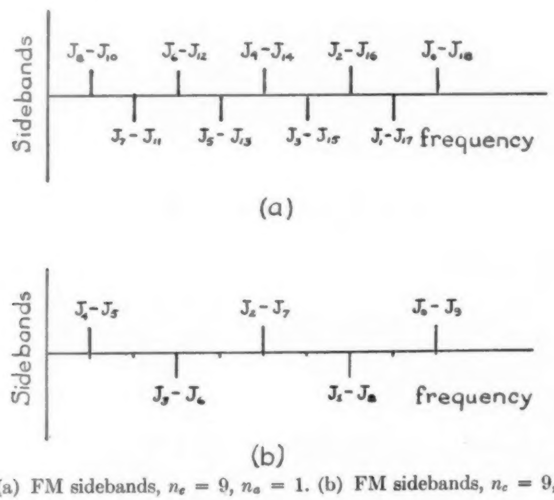

FIG. 2. (a) FM sidebands, $n_c = 9$, $n_a = 1$. (b) FM sidebands, $n_c = 9$, $n_a = 2$.

It may be seen that the ordinary expansion is inadequate for these particular cases, for it gives the correct frequency components but not the correct amplitude for each component.

c. *Multi-tone Modulation.* The same procedure applied to multi-tone modulation shows that for an AM wave of the form

$$e = A\left(1 + \sum_{s=1}^{k} m_s \cos \omega_s t\right) \cos \omega_c t, \tag{24}$$

there exist the Fourier coefficients

$$a_n = \frac{2A}{\pi} \int_0^{\pi} \left(1 + \sum_{s=1}^{k} m_s \cos \omega_{st}\right) \cos \omega_c t \cos n\omega_0 t \, d(\omega_0 t). \tag{25}$$

where ω_0 again is the highest common factor among ω_1, $\omega_2 \cdots \omega_k$, and each sideband can be identified as one of these Fourier harmonics.

For the FM case

$$e = A \sin \left(\omega_c t + \sum_{s=1}^{k} m_s \cos \omega_s t\right). \tag{26}$$

The Fourier coefficients,

$$a_n = \frac{2A}{\pi} \int_0^{\pi} \sin \left(n_c \omega_0 t + \sum_{s=1}^{k} m_s \cos n_s \omega_0 t\right) \cos n\omega_0 t \, d(\omega_0 t), \tag{27}$$

$$b_n = \frac{2A}{\pi} \int_0^{\pi} \sin \left(n_c \omega_0 t + \sum_{s=1}^{k} m_s \cos n_s \omega_0 t\right) \sin n\omega_0 t \, d(\omega_0 t), \tag{28}$$

can be obtained and one has,

$$a_n = \left[\sum_{k_{s2}}^{\pm\infty} \left(\prod_{s=1}^{K} J_{k_{s2}}(m_s)(-1)^{k_b - 1/2}\right) + \sum_{k_{s1}}^{\pm\infty} \left(\prod_{s=1}^{K} J_{k_{s1}}(m_s)(-1)^{(k_b - 1)/2}\right)\right] A, \tag{29}$$

where

$$\left|\sum_{s=1}^{K} k_{s1} n_s\right| = |\, n_c + n \,|, \qquad \left|\sum_{s=1}^{K} k_{s2} n_s\right| = |\, n_c - n \,|, \tag{29a}$$

and

$$k_b = \sum_{s=1}^{K} k_{s1} \quad \text{or} \quad \sum_{s=1}^{K} k_{s2}\,, \tag{29b}$$

when they are odd numbers.

$$b_n = \left[\sum_{k_{s1}}^{\pm\infty} \left(\prod_{s=1}^{K} J_{k_{s1}}(m_s)(-1)^{k_a/2}\right) - \sum_{k_{s2}}^{\pm\infty} \left(\prod_{s=1}^{K} J_{k_{s2}}(m_s)(-1)^{k_a/2}\right)\right] A \tag{30}$$

where

$$\left|\sum_{s=1}^{K} k_{s1} n_s\right| = |\, n_c - n \,|, \qquad \left|\sum_{s=1}^{K} k_{s2} n_s\right| = |\, n_c + n \,|, \tag{30a}$$

and

$$k_a = \sum_{s=1}^{K} k_{s1} \quad \text{or} \quad \sum_{s=1}^{K} k_{s2}\,, \tag{30b}$$

when they are even numbers.

Therefore, we have

$$e = \sum_n a_n \cos n\omega_0 t + \sum_n b_n \sin n\omega_0 t,$$

which is considerably different from the ordinary result,

$$e = \sum_{k_s=0}^{\pm\infty} \left\{ \prod_{s=1}^{K} J_{k_s}(m_s) \right\} \cos\left(\sum_{s=1}^{K} k_s \omega_s t \right). \tag{31}$$

d. *Modulation Products from a Linear Rectifier.* The subject of heterodyne detection has been investigated by many. Engineering practice assumes that detector can follow the envelope ideally so that higher harmonics can be neglected and the difference frequency taken as the fundamental. W. R. Bennett [3] gives a double Fourier series development of the output of such a rectifier for any amplitude and frequency ratio which seems to be the only exact analysis which has so far appeared. The present method of analysis appears to be an interesting and useful alternative.

Express Heaviside's Unit Function by,

$$1(t) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{\sin \omega t}{\omega} \, d\omega. \tag{32}$$

Assume an input wave of the form

$$e = A \cos \omega_1 t + B \cos \omega_2 t. \tag{33}$$

From (3) and (32) the output wave from a zero-bias linear rectifier is

$$\begin{aligned} e_0 &= \left\{ \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{\sin \omega e}{\omega} \, d\omega \right\} e \\ &= \frac{1}{2} (A \cos \omega_1 t + B \cos \omega_2 t) \\ &\quad + \frac{A \cos \omega_1 t + B \cos \omega_2 t}{\pi} \int_0^\infty \frac{\sin \omega (A \cos \omega_1 t + B \cos \omega_2 t)}{\omega} \, d\omega. \end{aligned} \tag{34}$$

Let $\omega_0 t = x$, where ω_0 is again the highest common factor between ω_1 and ω_2, such that $\omega_1 = n_1 \omega_0$, $\omega_2 = n_2 \omega_0$. Then the Fourier series of (34), of cosine terms only, can be specified completely by

$$\begin{aligned} a_n = \frac{2}{\pi} \int_0^\pi \Bigg[&\frac{1}{2} (A \cos n_1 x + B \cos n_2 x) \\ &+ \frac{A \cos n_1 x + B \cos n_2 x}{\pi} \int_0^\pi \frac{\sin (A \cos n_1 x + B \cos n_2 x)}{\omega} \, d\omega \Bigg] \cos nx \, dx. \end{aligned} \tag{35}$$

The first part of (35) gives

$$a_{n_1} = A/2, \qquad a_{n_2} = B/2. \tag{36}$$

The second part can be written as

$$\begin{aligned} I_2 = \frac{2}{\pi^2} \int_0^\infty \int_0^\pi \Bigg[&(A \cos n_1 x + B \cos n_2 x) \Bigg\{ \left(\sum_{K=0}^{\infty} \mathrm{E}_{2K+1} J_{2K+1}(A\omega)(-1)^K \cos (2K+1) n_1 x \right) \\ &\cdot (\mathrm{E}_{2K} J_{2K}(B\omega)(-1)^K \cos 2K n_2 x) + \left(\sum_{K=0}^{\infty} \mathrm{E}_{2K} J_{2K}(A\omega)(-1)^K \cos 2K n_1 x \right) \\ &\cdot \left(\sum_{K=0}^{\infty} \mathrm{E}_{2K+1} J_{2K+1}(B\omega)(-1)^K \cos (2K+1) n_1 x \right) \Bigg\} \Bigg] \cos nx \, dx \, \frac{d\omega}{\omega}, \end{aligned} \tag{37}$$

by using the expansions,

$$\cos (m \cos x) = \sum_{K=0}^{\infty} \mathrm{E}_{2K} J_{2K}(m)(-1)^K \cos 2Kx, \tag{38a}$$

$$\sin (m \cos x) = \sum_{K=0}^{\infty} \mathrm{E}_{2K+1} J_{2K+1}(m)(-1)^K \cos (2K + 1)x, \tag{38b}$$

where E_i is the Neumann E-factor defined as,

$$\begin{aligned} \mathrm{E}_i &= 1, \qquad i = 0, \\ \mathrm{E}_i &= 2, \qquad i \neq 0. \end{aligned} \tag{39}$$

There are in (37) infinitely many terms in the integrand, each term involving four cosine functions multiplied together associated with a product of two Bessel coefficients which are constant if we integrate with respect to x first. Integration with respect to x shows that each term in the integrand is not zero only when the sum and difference of two of the four cosine angles are equal respectively to the sum or difference of the other two angles. Following this the formula below is obtained.

If n_1 is even, and n_2 is odd, then for n odd

$$\begin{aligned} a_n = {} & \frac{A}{\mathrm{E}_n \pi} \int_0^{\infty} \sum_{S_1, K=0}^{\infty} \mathrm{E}_{2S_1+1}(-1)^{K+S_1} J_{2K}(A\omega) J_{2S_1+1}(B\omega) \frac{d\omega}{\omega} \\ & + \frac{B}{\mathrm{E}_n \pi} \int_0^{\infty} \sum_{S_2, K=0}^{\infty} \mathrm{E}_{2K+1}(-1)^{K+S_2} J_{2K+1}(A\omega) J_{2S_2}(B\omega) \frac{d\omega}{\omega} \end{aligned} \tag{40a}$$

where

$$2S_1 + 1 = \left| \frac{\pm n \pm n_1 \pm 2Kn_1}{n_2} \right|, \tag{40b}$$

$$2S_2 = \left| \frac{\pm n \pm n_2 \pm (2K + 1)n_1}{n_2} \right|, \tag{40c}$$

and for n even

$$\begin{aligned} a_n = {} & \frac{A}{\mathrm{E}_n \pi} \int_0^{\infty} \sum_{S_1, K=0}^{\infty} \mathrm{E}_{2K+1}(-1)^{K+S_1} J_{2K+1}(A\omega) J_{2S_1}(B\omega) \frac{d\omega}{\omega} \\ & + \frac{B}{\mathrm{E}_n \pi} \int_0^{\infty} \sum_{S_2, K=0}^{\infty} \mathrm{E}_{2S_2+1}(-1)^{K+S_2} J_{2K}(A\omega) J_{2S_2+1}(B\omega) \frac{d\omega}{\omega}, \end{aligned} \tag{41a}$$

where,

$$2S_1 = \left| \frac{\pm n \pm n_1 \pm (2K + 1)n_1}{n_2} \right|, \tag{41b}$$

$$2S_2 + 1 = \left| \frac{\pm n \pm n_2 \pm 2Kn_1}{n_2} \right|. \tag{41c}$$

If n_1 and n_2 are both odd, all odd harmonics are missing in the output because then in (42) there will be no possibility of combining the angles such that the whole integral

is not zero. For the even harmonics a_n is given by (41a) with (41b) and (41c) replaced by,

$$2S_1 = \left| \frac{\pm n \pm n_1 \pm (2K + 1)n_1}{n_2} \right|. \tag{42a}$$

and

$$2S_2 + 1 = \left| \frac{\pm n \pm n_2 \pm 2Kn_1}{n_2} \right|. \tag{42b}$$

In these formulas, the summations should run over all possible integral values of S and K that may satisfy the Diophantine equations in absolute value form.

Without loss of generality, we can assume that $A > B$ (The case $A = B$ will be discussed later). Then (40) and (41) can be integrated as a special case of the infinite discontinuous integral of Weber and Schafheitlin [4]. Thus if n_1 is even and n_2 odd, then for n odd,

$$\begin{aligned} a_n = \frac{A}{\mathrm{E}_n\pi} \sum_{S_1,K=0}^{\infty} \left(\frac{B}{A}\right)^{2S_1+1} & \frac{\Gamma\left(\frac{2K + 2S_1 + 1}{2}\right)(-1)^{K+S_1}}{\Gamma\left(\frac{2K - 2S_1 + 1}{2}\right)\Gamma(2S_1 + 2)} \\ & \cdot F\left\{\frac{2K + 2S_1 + 1}{2}, \frac{-2K + 2S_1 + 1}{2}, 2S_1 + 2, \frac{B^2}{A^2}\right\} \\ & + \frac{B}{\mathrm{E}_n\pi} \sum_{S_2,K=0}^{\infty} \left(\frac{B}{A}\right)^{2S_2} \frac{\Gamma\left(\frac{2K + 2S_1 + 1}{2}\right)(-1)^{K+S_2}}{\Gamma\left(\frac{2K - 2S_2 + 3}{2}\right)\Gamma(2S_2 + 1)} \\ & \cdot F\left\{\frac{2K + 2S_2 + 1}{2}, \frac{-2K + 2S_2 - 1}{2}, 2S_2 + 1, \frac{B^2}{A^2}\right\}. \end{aligned} \tag{43}$$

Here equations (40b) and (40c) have to be satisfied by $2S_2 + 1$ and $2S_2$.
For n even

$$\begin{aligned} a_n = \frac{A}{\mathrm{E}_n\pi} \sum_{S_1,K=0}^{\infty} \left(\frac{B}{A}\right)^{2S_1} & \frac{\Gamma\left(\frac{2K + 2S_1 + 1}{2}\right)(-1)^{K+S_1}}{\Gamma\left(\frac{2K - 2S_1 + 3}{2}\right)\Gamma(2S_1 + 1)} \\ & \cdot F\left\{\frac{2K + 2S_1 + 1}{2}, \frac{-2K + 2S_1 - 1}{2}, 2S_1 + 1, \frac{B^2}{A^2}\right\} \\ & + \frac{B}{\mathrm{E}_n\pi} \sum_{S_2,K=0}^{\infty} \left(\frac{B}{A}\right)^{2S_2+1} \frac{\Gamma\left(\frac{2K + 2S_1 + 1}{2}\right)(-1)^{K+S_2}}{\Gamma\left(\frac{2K - 2S_2 + 1}{2}\right)\Gamma(2S_2 + 2)} \\ & \cdot F\left\{\frac{2K + 2S_2 + 1}{2}, \frac{-2K + 2S_2 + 1}{2}, 2S_2 + 1, \frac{B^2}{A^2}\right\} \end{aligned} \tag{44}$$

where $2S_1$ and $2S_2 + 1$ satisfy (41b) and (41c).

If n_1 and n_2 are both odd, only even harmonics exist in the output and these are given by (44) with the Diophantine equations replaced by those of (42a) and (42b).

In the foregoing equations, $\Gamma(x)$ is the gamma function of argument x and $F(a, b, c, x)$ is the hypergeometric function of parameters a, b, c and argument x.

If $A = B$, (40) and (41) still exist, but the hypergeometric functions simplify to gamma functions, i.e.,

$$F(a, b, c, 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}. \tag{45}$$

This Fourier series analysis of the output shows two interesting results. First, it shows that the output may have important components of frequency lower than the difference frequency. When the difference frequency is not too much smaller than the beating frequencies and if it is not the highest common factor, there will be beat tones of considerable amplitude at frequencies lower than the difference frequency. Secondly, for a particular frequency component in the output, these formulas give the amplitudes of all components provided the Diophantine equations are solved for all the possible S and K. For illustration, suppose $f_1 = 800, f_2 = 1400$ such that the highest common factor is 200, and $n_1 = 4$, $n_2 = 7$. Then instead of the difference frequency 600, one would have the series of frequencies of values 200, 400, 600, 800, 1,000 etc. in the output. The frequencies 200 and 400 are both lower than the difference 600. On the other hand, suppose $f_1 = 210, f_2 = 330$, so that the highest common factor is 30 and $n_1 = 7$, $n_2 = 11$. We would expect the series of frequencies 30, 60, 90, 120, 150 etc. in the output. Here, however, since n_1 and n_2 are both odd, the odd harmonics 30, 90, 150 etc. will be missing except 210 and 330 which will be present, (Eq. 36).

e. *Arbitrary Wave-Shape Modulation.* It is evident that the same analysis for frequency modulation by any arbitrary wave-shape would exhibit the same reflected sideband phenomena as the case of sinusoidal modulation. A particular case of rectangular wave modulation has been analyzed in detail and the result compared with the ordinary spectrum to verify this conclusion. The result, however, does not seem to deserve more space here.

2. *Modulation Products With No Exact Period.* When a common factor among the component frequencies present in the modulation products does not exist, the wave as a whole is not periodic. The theory of almost periodic function then relieves us of any possible logical confusion. Here a definite spectrum still exists; the frequency components, however, are no longer related by a multiple of a common component but are determined by certain characteristic values as explained in connection with (8) and (9).

To illustrate the principle, consider again the amplitude-modulated wave $e = A(1 + m \text{ Cos } \omega_a t) \text{ Cos } \omega_c t$. If ω_c and ω_a have no factor in common, then the conventional Fourier analysis no longer applies. However by, (8),

$$\begin{aligned} a_n &= \lim_{T\to\infty} \frac{A}{T} \int_0^T (1 + m \cos \omega_a t) \cos \omega_c t \cos \lambda t \, dt \\ &= \lim_{T\to\infty} \frac{A}{T} \int_0^T [\cos (\omega_c + \lambda)t + \cos (\omega_c - \lambda)t] \, dt \\ &\qquad + \lim_{T\to\infty} \frac{mA}{2T} \int_0^T [\cos (\omega_c + \omega_a + \lambda)t + \cos (\omega_c + \omega_a - \lambda)t \\ &\qquad + \cos (\omega_c - \omega_a + \lambda)t + \cos (\omega_c - \omega_a - \lambda)t] \, dt. \end{aligned} \tag{46}$$

After integration each term in (46) will be of the form

$$\lim_{T\to\infty} \frac{A}{T} \cdot \frac{\sin (K \pm \lambda)t}{K \pm \lambda} \Bigg|_0^T, \tag{47}$$

and will be identically zero except when $\lambda = \pm K$. Therefore, since we are dealing with real or positive frequency only, the characteristic values are $\lambda_1 = \omega_c$, $\lambda_2 = \omega_c + \omega_a$, and $\lambda_3 = \omega_c - \omega_a$. It can easily be shown that

$$a_1 = A, \qquad a_2 = mA/2, \qquad a_3 = mA/2. \tag{48}$$

These are the same as the sideband amplitudes obtained in the periodic case.

The other kinds of modulation, which so far have been considered only for the case where the carrier and modulating frequencies have a common factor, may similarly be handled by this method for the almost periodic case. Also the results may be inferred for the case where ω_c and ω_m have a very small highest common factor and *almost* have a larger highest common factor. Here we would expect that the true modulation products would be large for the cases where they most nearly coincided with the products for the period based on the *large* "highest common factor".

IV. Findings on Energy Relationships. It is well known that in amplitude modulation the modulated wave has its energy increased by an amount corresponding to that in the sidebands. From the point of view of Fourier's series, this finding is nothing but an application of Parseval's theorem, since the energy or average power per cycle of an oscillation is proportional to the average square of the wave, and hence equal to the sum of the square of its Fourier coefficients. Thus, for the wave in equation (11) the energy E is,

$$E = \sum_{n=1}^{\infty} a_n^2 = A^2 + \frac{1}{4} A^2m^2 + \frac{1}{4} A^2m^2 = A^2 + \frac{1}{2} A^2m^2 \tag{49}$$

which agrees with the usual result.

For an *FM* wave, the coefficients obtained from (21), (i.e. when $n_a = 1$, n_c either even or odd) give

$$E = \sum_{n=1}^{\infty} b_n^2 = A^2 \sum_{n=1}^{\infty} [J^2_{n_c-n}(m_f) + J^2_{n_c+n}(m_f) - 2J_{n_c-n}(m_f)J_{n_c+n}(m_f)], \tag{50}$$

and the coefficients from (22), i.e. for $n_a = 2$, n_c odd, give

$$E = \sum_{n=1}^{\infty} b_n^2 = A^2 \sum_{K=0}^{\infty} [J^2_{n_c'-K}(m_f) + J^2_{n_c'+K+1}(m_f) - 2J_{n_c'-K}(m_f)J_{n_c'+K+1}(m_f)] \tag{51}$$

where $n_c' = (n_c - 1)/2$, and $2K + 1 = n$.

Equations (50) and (51) can be simplified (Appendix) to

$$E = A^2[1 - J_{2n_c}(2m_f)], \tag{52}$$

and

$$E = A^2[1 - J_{n_c}(2m_f)] \text{ respectively.} \tag{53}$$

The general case when $n_a \neq 1$ or 2 as in equation (23) can be easily analyzed by

using the fact that as n runs from 1 to ∞, the Fourier coefficients $J_i(m_f)$ will each occur twice when $i \neq 0$ and only once when $i = 0$. Therefore Parseval's formula gives, with the aid of the relation

$$J_0^2(m_f) + 2 \sum_{n=1}^{\infty} J_n^2(m_f) = 1,$$

$$E = A^2 \left[J_0^2(m_f) + 2 \sum_{n=1}^{\infty} J_n^2(m_f) \right] = A^2. \tag{54}$$

Since the energy before modulation is A^2, (52) and (53) show that in those two particular cases frequency modulation decreases the wave energy because $J_K(x)$ is always positive if $x < K$. In the case of (54), the energy remains unchanged.

Appendix

Equations (52) and (53) may be derived by starting with equation (50) in the text,

$$E = A^2 \sum_{n=\infty}^{1} [J_{n_c-n}^2(m_f) + J_{n_c+n}^2(m_f) - RJ_{n_c-n}(m_f)J_{n_c+n}(m_f)]. \tag{1'}$$

Let $n_c + n = m$, then $n_c - n = 2n_c - m$.
When $n = 1$, $m = n_c + 1$, and when $n = \infty$, $m = \infty$; therefore,

$$E = A^2 \sum_{m=n_c+1}^{\infty} [J_{2n_c-m}^2(m_f) + J_m^2(m_f) - 2J_{2n_c-m}(m_f)J_m(m_f)]. \tag{2'}$$

Now

$$\sum_{m=n_c+1}^{\infty} J_{2n_c-m}^2(m_f) = \sum_{m=n_c+1}^{2n_c} J_{2n_c-m}^2(m_f) + \sum_{m=2n_c+1}^{\infty} J_{2n_c-m}^2(m_f)$$

$$= \sum_{m=0}^{n_c-1} J_m^2(m_f) + \sum_{m=-1}^{-\infty} J_m^2(m_f), \tag{3'}$$

and

$$\sum_{m=n_c+1}^{\infty} J_m^2(m_f) = \sum_{m=0}^{\infty} J_m^2(m_f) - \sum_{m=0}^{n_c} J_m^2(m_f), \tag{4'}$$

so that

$$\sum_{m=n_c+1}^{\infty} [J_{2n_c-m}^2(m_f) + J_m^2(m_f)] = \sum_{m=-\infty}^{\infty} J_m^2(m_f) - J_{n_c}^2(m_f). \tag{5'}$$

Since $J_{-m}(m_f) = (-1)^m J_m(m_f)$, $\sum_{m=-\infty}^{\infty} J_m^2(m_f)$ can be written as

$$J_0^2(m_f) + 2 \sum_{m=1}^{\infty} J_m^2(m_f) = 1. \tag{6'}$$

Equation (5′) becomes, therefore,

$$\sum_{m=n_c+1}^{\infty} [J^2_{2n_c-m}(m_f) + J^2_m(m_f)] = 1 - J^2_{n_c}(m_f). \tag{7′}$$

The product sum

$$\sum_{m=n+1}^{\infty} 2J_{2n_c-m}(m_f)J_m(m_f)$$

can be treated as follows:

$$\sum_{m=n_c+1}^{\infty} 2J_{n_c-m}(m_f)J_m(m_f)$$

$$= \sum_{m=n_c+1}^{\infty} [J_{2n_c-m}(m_f)J_m(m_f)] + \sum_{m=n_c+1}^{\infty} J_{2n_c-n}(m_f)J_m(m_f)$$

$$= \left[\sum_{m=n+1}^{2n_c} J_{2n_c-m}(m_f)J_m(m_f) + \sum_{m=2n_c+1}^{\infty} J_{2n_c-m}(m_f)J_m(m_f)\right]$$

$$+ \left[\sum_{m=0}^{\infty} J_{2n_c-m}(m_f)J_m(m_f) - \sum_{m=0}^{n_c} J_{2n_c-m}(m_f)J_m(m_f)\right]$$

$$= \left[\sum_{m=0}^{n_c-1} J_m(m_f)J_{2n_c-m}(m_f) + \sum_{m=-1}^{-\infty} J_m(m_f)J_{2n_c-m}(m_f)\right]$$

$$+ \left[\sum_{m=0}^{\infty} J_{2n_c-m}(m_f)J_m(m_f) - \sum_{m=0}^{n_c} J_{2n_c-m}(m_f)J_m(m_f)\right]$$

$$= \sum_{m=-\infty}^{\infty} J_{2n_c-m}(m_f)J_m(m_f) - J^2_{n_c}(m_f). \tag{8′}$$

Combining (7′) and (8′) one has (1′) in the simple form

$$E = A^2\left[1 - \sum_{m=-\infty}^{\infty} J_{2n_c-m}(m_f)J_m(m_f)\right]. \tag{9′}$$

By the Addition Theorem of Neumann and Schlafli (4)

$$J_m(y + z) = \sum_{n=-\infty}^{\infty} J_n(y)J_{n-m}(z). \tag{10′}$$

If $y = z$, therefore, $J_n(2y) = \sum_{n=-\infty}^{\infty} J_m(y)J_{n-m}(y)$ and (9′) can be replaced by

$$E = A^2[1 - J_{2n_c}(2m_f)] \tag{11′}$$

which is equation (52).

By an entirely similar process, it can be shown that (51) can be simplified to

$$E = A^2[1 - J_{n_c}(2m_f)] \tag{12′}$$

which is equation (53).

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TWO NEW NON-LINEARIZED CONICAL FLOWS*

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1. Introduction. A steady compressible non-viscous flow is *conical* if it contains a *vertex* P, such that on every half line through P the velocity components, pressure, density, and entropy are constant. Various linearized conical flows have been discussed by numerous authors. However, only three examples of non-linearized conical potential flow fields are known to exist mathematically: Prandtl-Meyer flow around an edge [4][1] with or without sweep; Taylor-Maccoll flow about a non-yawing circular cone [5]; and an axisymmetric flow through a convergent nozzle discussed by Busemann [1]. In this paper the construction of two new examples will be considered. Both contain two regions of swept Prandtl-Meyer flow. In the first, the boundary has been chosen to prevent them from interacting, and the hodograph is one-dimensional. From it can be obtained a flow, with attached plane shock, over an object resembling an airplane with a swept-forward wing of positive dihedral and with a thick fin. In the second, the boundary has been chosen to permit interaction and the hodograph is two-dimensional. It was studied originally in the hodograph space by one of the authors [2]. In the present treatment, the need to consider possible difficulties in constructing the flow field from a knowledge of the hodograph has been avoided by confining the discussion to the physical space. It should also be remarked that in both examples the second order partial differential equation for conical potential flow is of mixed type.

A pair of these examples could conceivably be used to study a particular, atypical case of wing-body interference. Numerical results can easily be calculated, if necessary, with the aid of a characteristics table and by means of standard techniques for numerical integration of ordinary differential equations and numerical solution of characteristic initial value problems for second order hyperbolic partial differential equations in two independent variables.

2. Fundamental Ideas. The velocity potential function of a steady irrotational non-viscous isentropic flow satisfies

$$(a^2\delta_{ij} - u_iu_j)\partial^2\varphi/\partial x_i\,\partial x_j = 0 \tag{2.1}$$

where x_i $(i = 1, 2, 3)$ are rectangular coordinates,

$$u_i = \partial\varphi/\partial x_i \tag{2.2}$$

is the velocity component parallel to the x_i-axis in units of the maximum speed of flow,

$$a^2 = 1/2(\gamma - 1)(1 - u_iu_i) = 1/2(\gamma - 1)(1 - q^2) \tag{2.3}$$

is the square of the velocity of sound, Kronecker's delta $\delta_{ij} = 1(0)$ if $i = (\neq)\, j$, and the convention has been adopted that repeated indices imply summation over their range. If $q^2 = u_iu_i > a^2$, the flow is supersonic, and there exist real characteristic surfaces, which are envelopes of the Mach cones

$$[u_i(x_i^* - x_i)]^2 = (q^2 - a^2)(x_i^* - x_i)(x_i^* - x_i) \tag{2.4}$$

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[1]Numbers in brackets designate references listed at the end of the paper.

where x_i^* are running coordinates. At these surfaces the partial derivatives of φ of the second or higher order may have discontinuities while φ and $\partial\varphi/\partial x_i$ remain continuous, as may occur when two solutions are patched together.

The image of a flow under the mapping $x_i \to u_i$ is defined to be its hodograph. In general, three-dimensional flows have three-dimensional hodographs, but not the examples in the following sections. Flows with one (two)-dimensional hodographs have been called *simple* (*double*) waves. The relevant properties of simple waves, discussed in [3], will be summarized briefly. For some function $\mu(x)$, $u_i = u_i(\mu)$. Now (2.1) implies

$$a^2 u_i' u_i' = (u_i u_i')^2 = (qq')^2 \tag{2.5}$$

where $u_i' = du_i/d\mu$. The hodograph is any curve obtainable from a Prandtl-Meyer epicycloid by deforming its plane into a cone with vertex at $u_i = 0$. To specify the curve completely an additional equation and initial conditions are required. For example, in a swept Prandtl-Meyer flow the velocity component parallel to a fixed unit vector λ_i must be constant, that is,

$$u_i \lambda_i = \text{constant}. \tag{2.6}$$

In the physical space the *prototype* of the hodograph point $u_i(\mu)$, a surface μ = constant which also bears constant pressure, density, and entropy, is a plane normal to u_i'. The exact locations of the prototype planes in the physical space are as yet undetermined. If all are forced to pass through a common point, the simple wave will be a conical flow. Finally, note that a prototype plane is a characteristic surface, all of whose Mach cones are congruent and have parallel axes.

Now let

$$X_\alpha = x_\alpha/z(\alpha = 1, 2), \qquad z = x_3, \qquad w = u_3 \tag{2.7}$$

Later it will also be convenient to use $X = X_1$, $Y = X_2$, $u = u_i$, and $v = u_2$. Note that to a point (curve) in the X_1X_2-plane there corresponds in the x_i-space a line through (cone with vertex at) the origin. Consider a general conical flow with vertex at the origin. Since $u_i = \partial\varphi/\partial x_i$ are homogeneous of degree zero in the x, s, φ may be assumed with no loss of generality to be homogeneous of degree one. Then for some Φ

$$\varphi = z\Phi(X_1, X_2) \tag{2.8}$$

$$u_\alpha = \partial\varphi/\partial x_\alpha = \partial\Phi/\partial X_\alpha, \qquad w = \partial\varphi/\partial z = \Phi - X_\alpha u_\alpha \tag{2.9}$$

$$z\, \partial^2\varphi/\partial x_\alpha\, \partial x_\beta = \partial^2\Phi/\partial X_\alpha\, \partial X_\beta, \qquad z\, \partial^2\varphi/\partial x_\alpha\, \partial z = -X_\beta\, \partial^2\Phi/\partial X_\alpha\, \partial X_\beta$$

$$z\, \partial^2\varphi/\partial z^2 = X_\alpha X\, \partial^2\Phi/\partial X_\alpha\, \partial X \tag{2.10}$$

and (2.1) becomes

$$[a^2(\delta_{\alpha\beta} + X_\alpha X_\beta) - (u_\alpha - wX_\alpha)(u_\beta - wX_\beta)]\, \partial^2\Phi/\partial X_\alpha\, \partial X_\beta = 0. \tag{2.11}$$

Characteristics for (2.11) are curves $X_\alpha = X_\alpha(t)$ in the X_1X_2-plane on which the partial derivatives of Φ of second or higher order may have discontinuities, while Φ and $\partial\Phi/\partial X_\alpha$ remain continuous. To these curves there correspond cones $x_\alpha/z = X_\alpha(t)$ on which, by (2.10) or its analogs the partial derivatives of φ of second or higher order will have corresponding discontinuities, while by (2.8) and (2.9) φ and $\partial\varphi/\partial x_i$ remain continuous.

Thus $x_\alpha/z = X_\alpha(t)$ are conical characteristic surfaces for (2.1). A plane tangent to one of them along the ray $x_\alpha/z = X_\alpha$ must be tangent to the Mach cone (2.4) based on $u_i = u_i(X_1, X_2)$ with vertex at the origin and must correspond to the tangent to the characteristic $X_\alpha = X_\alpha(t)$. Hence the characteristic directions for (2.11) must be those of the lines through (X_1, X_2) tangent to the conic

$$(u_\alpha X_\alpha^* + w)^2 = (q^2 - a^2)(X_\alpha^* X_\alpha^* + 1) \tag{2.12}$$

with running coordinates X_α^*. The type of (2.11) is hyperbolic, parabolic, or elliptic accordingly as (X_1, X_2) is outside, on, or inside (2.12). For a conical simple wave (2.11) is of hyperbolic type (except possibly on a curve on which the type is parabolic), with straight characteristics. Furthermore, a region of hyperbolic type adjacent to a region of uniform flow must be a conical simple wave.

3. Flow along a conical wall. It is well known that swept Prandtl-Meyer flow over a dihedral angle can be generalized into simple wave flow past a curved cylindrical wall. Now a generalization to simple wave flow in a trough with boundaries composed of plane and conical segments will be discussed.

Consider uniform supersonic flow along a plane wall on which l is a line inclined with respect to the direction of flow by more than the Mach angle. Let P be any point on l. Extend the boundary beyond l as a cone through l with vertex P, and eventually join the cone to another plane segment. Attempt to fit a conical simple wave to this boundary. Let the origin of coordinates be at P. Describe the boundary by

$$x_i = rv_i(\mu) \tag{3.1}$$

where r and μ are independent. With no loss of generality assume

$$v_i v_i = 1, \qquad v_i' v_i' = 1. \tag{3.2}$$

Then

$$v_i v_i' = v_i' v_i'' = 0, \qquad v_i v_i'' = -1. \tag{3.3}$$

Assume that the prototype planes pass through the rulings of (3.1). Then

$$u_i' v_i = 0 \tag{3.4}$$

and the boundary condition on the cone becomes $u_i = Av_i + Bv_i'$ for some scalar functions $A(\mu)$ and $B(\mu)$. By (3.3) and (3.4) $B = A'$, so

$$u_i(\mu) = A(\mu)v_i(\mu) + A'(\mu)v_i'(\mu). \tag{3.5}$$

Now (2.5) becomes $a^2A'^2(v_i''v_i'' - 1) = (A'^2 - a^2)\,(A + A'')^2$, where

$$a^2 = \frac{1}{2}(\gamma - 1)(1 - A^2 - A'^2). \tag{3.6}$$

For any curve on the unit sphere the curvature $v_i''\, v_i'' \geqq 1$. Also, as in Prandtl-Meyer flow with leading edge $v_i(\mu)$, the normal component of velocity $A' \geqq a \geqq 0$. Assume that the simple wave is an expansion. Then $A'\,(A + A'') \geq 0$, and

$$aA'(v_i''v_i'' - 1)^{1/2} = (A + A'')(A'^2 - a^2)^{1/2}. \tag{3.7}$$

The initial values of A and A' depend, of course, on the original velocity and orientation of l.

The prototype planes through the origin satisfy

$$u_i'(\mu)x_i = 0. \tag{3.8}$$

Join the velocity field (3.5), (3.8) to uniform flows at both ends. Let s be any streamline not on (3.1) which does not intersect the envelope of the prototype planes, defined by

$$x_i = rn_i(\mu), \qquad n_i n_i' = n_i u_i'' = 0, \qquad n_i n_i = 1. \tag{3.9}$$

Such streamlines exist if (3.1) and (3.9) do not intersect. As the second wall of the trough, also composed of plane and conical segments, choose the cone through s with vertex P. A special example of this type of flow can easily be derived from a Prandtl-Meyer flow. Retain as one side of the trough the plane walls of the original boundary, and use a conical stream sheet for the other wall.

4. A conical simple wave. Insert a half plane into a uniform supersonic flow at a moderate angle of attack, and make the angle between the leading edge and the undisturbed velocity greater than the Mach angle. On one side there will be a swept Prandtl-Meyer flow around the leading edge. On the other there will be an attached plane shock wave behind which there will be uniform flow (supersonic if the angle of attack is not too large) parallel to the half plane. Throughout the entire flow the component of velocity parallel to the leading edge is constant. Introduce a coordinate system with origin on the leading edge, z-axis parallel to the undisturbed velocity, yz-plane parallel to the uniform flow behind the shock, and hence normal to the shock. Discard that part of the flow on the side of the yz-plane that contains the downstream half of the leading edge. Reflect the remainder with respect to the yz-plane. So far the boundary, shown in Fig. 1 together

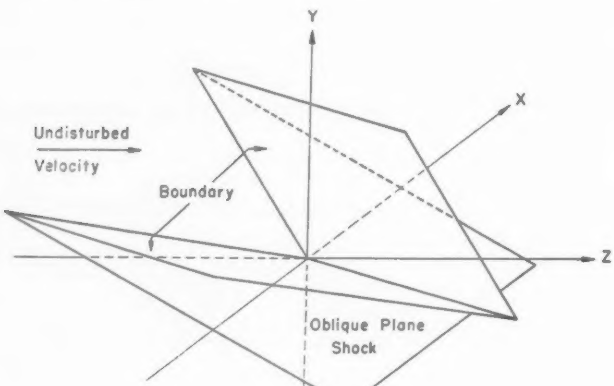

Figure I Shock and Tentative Boundary

with the shock, consists of a dihedral angle with congruent sectors removed from each face. Since the two halves of the shock are coplanar, the uniform flows on the compression

side join continuously. On the expansion side u_1 is double valued in the yz-plane. This difficulty can be avoided by modifying the upper side of the boundary as follows.

First, examine the hodograph of the swept Prandtl-Meyer flow. In Fig. 2 let ON^*

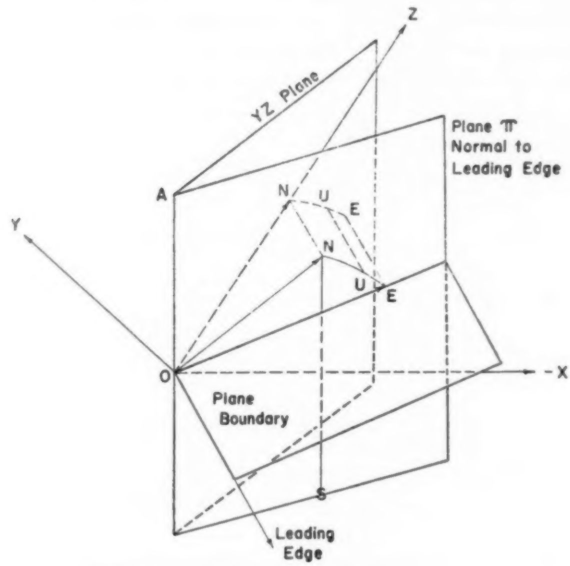

Figure 2 Hodograph of Swept Prandtl-Meyer Flow

be the undisturbed velocity in the yz-plane, and let N be the projection of N^* onto the plane Π that passes through the origin and is normal to the leading edge. During the expansion, the component of velocity normal to Π remains equal to NN^* while the component parallel to Π traces an arc NE of a Prandtl-Meyer epicycloid, shrunk by a factor $(1 - NN^{*2})^{1/2}$. Fig 1 shows that the x-axis is under the plane boundary, so the angle AOE is acute. Hence the epicycloidal arc NE cannot have another intersection with the line NS, parallel to OA. Now construct the hodograph by subjecting every point of NE to the displacement NN.* Clearly, during the expansion from ON^* to OE^* the angle between the velocity vector OU^* and the yz-plane (Π) steadily increases (decreases).

Now let Λ be the intersection of $x = 0$ and the plane Mach surfaces that pass through the leading edges and are based on the undisturbed velocity. Let P be any point on Λ. From the nature of the hodograph it is clear that the streamline through P for the simple wave in $x \geqq 0$ turns immediately into $x > 0$ and stays there. Hence an entire conical stream sheet through Λ bends into $x > 0$. Thus it is possible to separate the two regions of swept Prandtl-Meyer flow by means of a symmetrical conical fin, the thickness of which increases with increasing sweep. Note that at the junctions of the fin and the original boundary u_i is parallel to x_i . Accordingly, near the corresponding points X_α (2.11) is of elliptic type. On the other hand, for very large values of X_α near the boundary (2.11) is of hyperbolic type.

To obtain from this boundary a finite obstacle resembling an airplane, symmetrically terminate the wing, as in Fig. 3, at a trailing edge which is supersonic with respect to the uniform flows adjacent to both sides of the wing. On the lower side there will be an expansion around the trailing edges. The flow field just described will be unaffected up to the first Mach surface in the expansion fan emanating from the trailing edges. In particular, the leading edge shock will cease to be plane at its intersection with this surface. On the upper side there will be a shock attached to the trailing edges, ahead of which

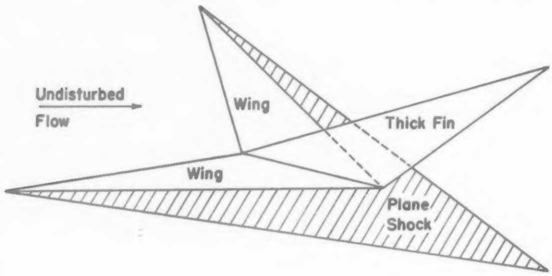

Figure 3 Schematic Representation of Finite Obstacle in Simple Wave Flow

the original flow will be unaltered. If the trailing edges are straight, as in Fig. 3, the shock and immediately following flow for either half of the wing will be conical with respect to the corresponding wing tip. Possible trailing edges for the fin would be its intersections with the trailing edge shocks. If a thick wing is desired, the upper surfaces need not be parallel to the lower surfaces. Finally the upper surfaces need not even be plane, but may be cylindrical or even conical, with vertices at the wing tips. However, after such changes the simple wave flows cease to be conical with respect to the origin.

5. Interaction of simple waves [2]. Return to the stage of the discussion at the end of the first paragraph of Section 3. Examination of Fig. 1 shows that cross sections by the planes $z = \pm 1$ would have the appearances of Figs. 4 and 5. E_R and E_L are traces

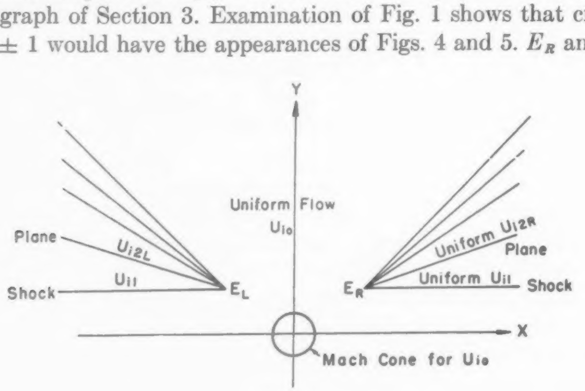

Figure 4 Trace by Z = -1

of the leading edges. The traces of the boundary and shock wave have been shown mainly to clarify the preceding example. They really have no immediate bearing on the interaction example, since it will be necessary to consider the possibility of having to reduce the amount of expansion and the extents of the interacting simple waves. The

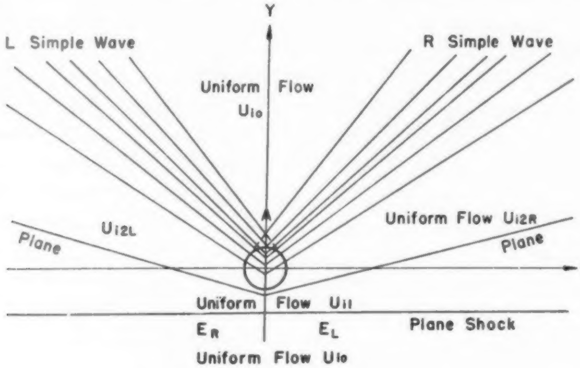

Figure 5 Trace of Tentative Flow by Z = + 1

boundary will actually be constructed later. The essential point is that the discussion starts from simple waves in which all of the straight characteristics pass through the centers E_R or E_L .

In Fig. 4 the simple waves have not begun to interact so the boundary need not be altered for $z < 0$. In Fig. 5 they begin to influence each other along the characteristics AB_R and AB_L , shown in a magnified view in Fig. 6. Since the type of (2.11) is hyperbolic

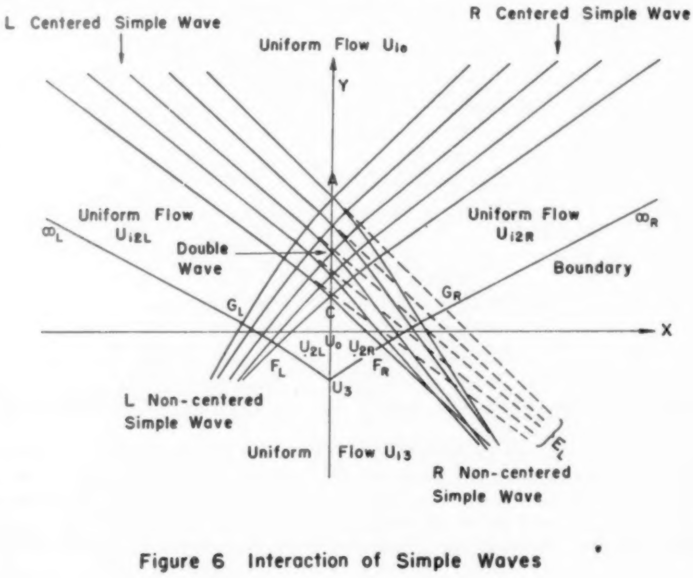

Figure 6 Interaction of Simple Waves

at A, the characteristics actually extend some distance beyond A as shown. In the simple waves u_α and w are known on AB_R and AB_L. Hence Φ and $\partial\Phi/\partial X_\alpha$ are known by (2.9). The characteristic initial value problem for (2.11) with these initial data will have a unique solution in some sufficiently small characteristic quadrilateral AB_RCB_L, where C may be assumed to be on the Y-axis. Since the initial data are symmetrical with respect to the Y-axis, so is Φ. Extend Φ beyond B_RC as the velocity potential of a conical simple wave. In general, the corresponding straight characteristics will not be centered. However, if the double wave AB_RCB_L is kept small enough, their envelope and intersections can be kept arbitrarily close to E_L, well beyond the boundary to be constructed later. Between the two simple waves in $X > 0$ there falls a region of uniform flow with the velocity u_{i2R} of B_R. Extend the definition of Φ symmetrically with respect to the Y-axis. Finally, between the two non-centered simple waves there falls a region of uniform flow with the velocity u_{i3} of C.

It remains to choose a boundary in $z > 0$. First let U_0, U_{2R}, U_{2L}, and U denote the *points* where rays from (0, 0, 0) parallel to u_{i0}, u_{i2R}, u_{i2L}, and u_{i3} intersect $z = 1$. Let $G_R\infty$, starting from the right edge of the right hand non-centered simple wave, be on the line E_RU_{2R}. Then u_{i2R} is tangent to the corresponding plane through the origin. Let F_RG_R be an integral curve of the equation $dX/(u - wX) = dY/(v - wY)$ of conical stream sheets. The final part of the boundary in $X > 0$ consists of the segment F_RU_3 which corresponds to a plane to which u_i is tangent. Extend the boundary symmetrically into $X < 0$.

Note that near U_3 (2.11) is of elliptic type.

It is interesting to observe that an alternation of double waves, simple waves, and regions of uniform flow similar to that in Fig. 6 also appears in an intersection of simple waves which occurs when uniform plane supersonic flow expands into an infinite sector. This is shown schematically in Fig. 7.

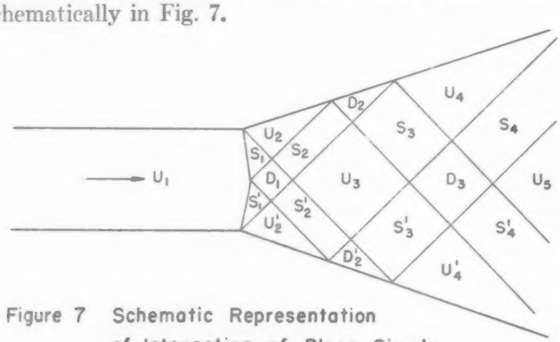

Figure 7 Schematic Representation of Interaction of Plane Simple Waves in Divergent Channel

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—NOTES—

INDUCED MASS WITH VARIABLE DENSITY*

By GARRETT BIRKHOFF (*Harvard University*)

The concept of induced mass, and some of its properties, are extended to the case of an incompressible fluid of variable density. The proofs parallel closely those given recently by the author, for the case of free boundaries[1].

1. Minimum principle. Let the fluid, supposed originally at rest and occupying a region R, be given an acceleration $\mathbf{a}$, by the motion of a wall W bounding the fluid internally. By continuity,

$$a_n = f(\mathbf{x}) \text{ on } W, \tag{1}$$

where $f(\mathbf{x})$ is the normal acceleration of W.

THEOREM 1. The acceleration kinetic energy

$$T = \frac{1}{2}\int_R \rho(\mathbf{a}\cdot\mathbf{a})\, dR \tag{2}$$

is minimized, relative to all other volume conserving flows satisfying (1).

Proof. Let $\mathbf{a} + \mathbf{b}$ be any other volume-conserving initial acceleration satisfying (1). Then Div $\mathbf{b} = 0$, and, by (1), $b_n = 0$ on W. Consider next the expansion

$$T = \frac{1}{2}\int_R \rho(\mathbf{a} + \mathbf{b})\cdot(\mathbf{a} + \mathbf{b})\, dR$$

$$= T_0 + \int_R (\rho\mathbf{a})\cdot\mathbf{b}\, dR + \frac{1}{2}\int_R \rho(\mathbf{b}\cdot\mathbf{b})\, dR.$$

Since the last term is positive unless $\mathbf{b}$ vanishes identically, it is sufficient to show that the middle integral is zero. But, since Div $\mathbf{b} = 0$ and (by the equations of motion, neglecting gravity[2]) $\rho\mathbf{a} = -\nabla p$,

$$\text{Div}\,(p\mathbf{b}) = p\,\text{Div}\,\mathbf{b} + (\nabla p)\cdot\mathbf{b} = -\rho\mathbf{a}\cdot\mathbf{b},$$

where p is the scalar pressure. Hence, by the Divergence Theorem, letting b_n denote the outward normal component of $\mathbf{b}$,

$$\int_R (\rho\mathbf{a})\cdot\mathbf{b}\, dR = \int_W p b_n\, dS = 0,$$

since $b_n = 0$ on W. (The same conclusion will hold if the boundary is partly "free", since then we can take $p = 0$.) This completes the proof.

The case of a free boundary (say, of an internal cavity) should not, of course, be confused with the boundary of an incompressible region of zero density.

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[1] "Induced mass with free boundaries," this *Quarterly* **10**, 81–86 (1952).

[2] Gravity can be neglected, for very rapid ("impulsive") accelerations; viscosity is without effect in the case of initial acceleration from rest.

2. Induced mass tensor and momentum. Letting $\mathbf{a}^1$, $\mathbf{a}^2$, $\mathbf{a}^3$ and $\mathbf{a}^4$, $\mathbf{a}^5$, $\mathbf{a}^6$ be the acceleration fields associated with *unit* infinitesimal translations and rotations, one can define, as previously[1], the symmetric 6×6 induced mass tensor (matrix) $\| T_{hk} \|$ by

$$T_{hk} = \int_R \rho \mathbf{a}^h \cdot \mathbf{a}^k \, dR = T_{kh} . \tag{3}$$

Since the diagonal components T_{hh} satisfy Theorem 1, we obtain immediately

COROLLARY 1. The diagonal components of induced mass are increased, if the fluid density is increased in any region (and left unchanged elsewhere).

Also, much as before, one can prove

COROLLARY 2. Let $\| T_{hk} \|$ and $\| T'_{hk} \|$ be associated with solids, of which the second is obtained from the first by replacing a mass Δm of fluid by solid. Then the (scalar) components of translation induced mass T_{hh} satisfy the inequality

$$T_{hh} \geqq T'_{hh} - \Delta m. \tag{4}$$

Again, letting p_h denote the initial pressure required to produce the initial acceleration $\mathbf{a}^h$, we have

$$-T_{hk} = \int_R (\nabla p_h) \cdot \mathbf{a}^k \, dR = \int_R \text{Div } (p_h \mathbf{a}^k) \, dR,$$

since $\rho \mathbf{a}^h = \nabla p_h$ by the equations of motion, and

$$\text{Div } (p_h \mathbf{a}^k) = p_h \cdot \text{Div } \mathbf{a}^k + \nabla p_h \cdot \mathbf{a}^k = \nabla p_h \cdot \mathbf{a}^k,$$

since the flow is incompressible ($\text{Div } \mathbf{a}^k = 0$).

By the Divergence Theorem,

$$-T_{hk} = \int_W p_h a_n^k \, dS, \tag{5}$$

where a_n^k is the normal component of $\mathbf{a}^k$, and hence (by continuity) the k^{th} direction cosine (for translation), and a similar moment producing factor for rotation if $k = 1$, 2, 3. Thus, in this case, T_{hk} is the total *thrust* in the k-direction, produced by the initial h-acceleration. Similarly, if $k = 4, 5, 6$, T_{nk} is the *moment* about the appropriate axis, produced by the h-acceleration. We conclude

THEOREM 2. The tensor component T_{hk} represents the total h-component of initial *pressure force* required to produce a unit k-acceleration of the missile.

The arguments given before[1] apply without change to prove also, if gravity is negligible, and if an "h-curve" is defined as before[1] to be a cylinder of finite cross-section bounded by parallels to the h-axis, in case $h = 1, 2, 3$, and as a circle perpendicular to, and centered on, the $(h - 3) =$ axis if $h = 4, 5, 6$,

THEOREM 3. Let a rigid solid be given a unit k-acceleration from rest, in an incompressible liquid of variable density. Then the initial rate of increase in the h-component of liquid momentum is T_{hk} , in any region C bounded by h-curves which contains the solid.

It is worth noting that, if the density $\rho(\lambda)$ is constant on each sphere of a concentric family, or on each ellipsoid of a confocal family, then the acceleration fields can be expressed as solutions of an ordinary differential equation. The details will be published elsewhere.

A NOTE ON THE APPLICATION OF SCHWINGER'S VARIATIONAL PRINCIPLE TO DIRAC'S EQUATION OF THE ELECTRON*

By H. E. MOSES (*Institute for Mathematics and Mechanics, New York University*)

Schwinger's variational principle has been used for a wide variety of problems involving wave motion in which it is desired to find the amplitude of a scattered wave in terms of the incident wave.** Schwinger's method makes use of the fact that the amplitude of the scattered wave satisfies a variational principle. We shall indicate this variational principle briefly.

Let us consider a vector space. We may define two different inner products, the Hermitian and symmetric inner product, in this vector space. The Hermitian inner product (a,b) of two vectors a and b is defined by the condition that

$$(a, b) = (b, a)^* \tag{1}$$

where the asterisk indicates the complex conjugate. The symmetric inner product is defined by the condition

$$(a, b) = (b, a). \tag{2}$$

Let us consider a vector space with a Hermitian inner product and consider a pair of equations

$$\begin{cases} a = Ky, \\ a' = K'y', \end{cases} \tag{3}$$

where K and K' are Hermitian adjoint operators which by definition satisfy the condition

$$(K'u, v) = (u, Kv) \tag{4}$$

for any two vectors u and v. If $K' = K$, K is a Hermitian operator. It can be shown from (3) that

$$(a', y) = (y', a) \tag{5}$$

which is called the *reciprocity theorem*. Let us define a number λ by

$$\lambda = \frac{1}{(y', a)} = \frac{1}{(a', y)}, \tag{6}$$

and the functional $\lambda\{v,v'\}$ by

$$\lambda\{v, v'\} = \frac{(v', Kv)}{(v', a)(a', v)} = \frac{(K'v', v)}{(v', a)(a', v)} \tag{7}$$

so that

$$\lambda\{y, y'\} = \lambda. \tag{8}$$

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*Received March 25, 1952.

**See, for example, Schwinger's unpublished nuclear physics notes, or the lectures of N. Marcuvitz in the notes "Recent Developments in the Theory of Wave Propagation", Inst. for Math. and Mech., N.Y.U., 1949–50. The problem of the present paper is treated abstractly in the first set of notes. The point of view of the present note is close to that of the latter set of notes.

It can be shown that $\lambda\{v,v'\}$ is stationary for independent variations of v and v' about the values y and y' respectively and that, therefore, from (8), the stationary value of $\lambda\{v,v'\}$ is λ.

A similar statement holds if the symmetric inner product is used. In this case K' is said to be the symmetric adjoint operator of K if K' and K are related by (4).

In Dirac's theory of the electron, the elements of the vector space are functions $f(x,\gamma)$ of the coordinates denoted collectively by the vector x, and of a variable γ which is restricted to four values which may be taken as 1, 2, 3, 4. These functions are called spinor components. The Hermitian inner product of $f(x,\gamma)$ and $g(x,\gamma)$ is given by

$$\sum_{\gamma=1}^{4} \int f(x, \gamma)^* g(x, \gamma)\, dx$$

and the symmetric inner product by

$$\sum_{\gamma=1}^{4} \int f(x, \gamma) g(x, \gamma)\, dx.$$

Dirac's wave equation for the electron in an electromagnetic field is

$$\frac{i\partial\psi(x, \gamma; t)}{\partial t} = H\psi(x, \gamma; t) \tag{9}$$

where H is an operator which operates on x and γ and is given by

$$H = H_0 + q \tag{10}$$

with

$$\left.\begin{aligned} H_0 &= \sum_{j=1}^{3} i\alpha_j \frac{\partial}{\partial x_j} - m\beta \\ q &= \sum_{j=1}^{3} \alpha_j A_j(x) + e\phi(x) \end{aligned}\right\}. \tag{11}$$

Here α_j and β are Hermitian operators which operate with respect to the variable only. They satisfy the following commutation relations

$$\left.\begin{aligned} \alpha_j\alpha_i + \alpha_i\alpha_j &= 2\delta_{ij}I, \\ \beta\alpha_j + \alpha_j\beta &= 0, \\ (\beta)^2 &= I, \end{aligned}\right\} \tag{12}$$

where I is the identity operator.

The operators α_j , β can be expressed as integral operators with kernels $\alpha_i(\gamma,\gamma')$, $\beta(\gamma,\gamma')$ which are the well-known Dirac matrices.

We have taken $h/2\pi = c = 1$. The mass of the electron is m and its charge is e. The functions $A_i(x)$ and $\phi(x)$ are the vector and scalar potentials of an electromagnetic field and are taken as real and are assumed to vanish if $|x| > r_0$, for some r_0.

We shall look for solutions of equation (9) which can be written as

$$\psi(x, \gamma; t) = e^{-iEt}\chi(x, \gamma; E) \tag{13}$$

so that equation (9) leads to

$$H\chi(x, \gamma; E) = E\chi(x, \gamma; E). \tag{14}$$

We shall write

$$\chi = \chi_{in} + \chi_{sc}, \tag{15}$$

and require that χ_{in} be a solution of

$$H_0\chi_{in}(x, \gamma; E) = E\chi_{in}(x, \gamma; E) \tag{16}$$

where H_0 is given by (11). Suitable solutions are the "spinor plane wave" solutions which have the form

$$\chi_{in}(x, \gamma; E) = \chi(\gamma; E, \eta, \tau)\frac{e^{i|k|(\eta x)}}{(2\pi)^{3/2}} \tag{17}$$

where η is a unit vector which specifies the direction of propagation, (ηx) is the inner product of the vectors x and η, and $|k|$ is the absolute value of the momentum vector and is given by the relation

$$|k|^2 = E^2 - m^2. \tag{18}$$

Here τ is a variable which is restricted to two values which may be taken as $+1$ and -1. The significance of τ is that it represents the component of the spin in the direction of the momentum.

By substituting (31) into the equation $H_0\chi_{in} = E\chi_{in}$ it is seen that the functions $\chi(\gamma; E, \eta, \tau) \equiv \chi(\gamma; E)$ satisfy the following equation

$$\left\{E + |k| \sum_{j=1}^{3} \alpha_j\eta_j + \beta m\right\}\chi(\gamma; E, \eta, \tau) = 0. \tag{19}$$

For the purpose of the present note, it is not necessary to give an explicit form for the functions $\chi(\gamma; E)$; these functions can be found in textbooks. However, it will be useful to indicate the orthogonality properties of the functions. These orthogonality relations are

$$\left.\begin{aligned}\sum_{\gamma}\chi(\gamma; E, \eta, \tau)^*\chi(\gamma; E, \eta, \tau) &= \delta_{\tau\tau'}\\ \sum_{\gamma}\chi(\gamma; -E, \eta, \tau)^*\chi(\gamma; E, \eta, \tau') &= 0,\end{aligned}\right\} \tag{20}$$

also

$$\sum_{\tau}\chi(\gamma; E, \eta, \tau)\chi(\gamma'; E, \eta, \tau) + \sum_{\tau}\chi(\gamma; -E, \eta, \tau)\chi(\gamma'; -E, \eta, \tau)^* = \delta(\gamma, \gamma'). \tag{21}$$

From (14), (15) and (16) it is seen that χ_{sc} satisfies

$$[E - H_0]\chi_{sc}(x, \gamma; E) = q\chi(x, \gamma; E) \tag{22}$$

the solution of which can be written in terms of influence function $g(x,\gamma; x', \gamma')$,

$$\chi_{sc}(x, \gamma; E) = \sum_{\gamma'}\int g(x, \gamma; x', \gamma')q\chi(x', \gamma'; E) \tag{23}$$

where

$$[E - H_0]g(x, \gamma; x', \gamma') = \delta(x - x')\ \delta(\gamma, \gamma'). \tag{24}$$

In (24) the operator $[E - H_0]$ operates on the variables x, γ. Since the physically interesting problem is that for which χ_{sc} is an outgoing wave, we shall take an outgoing wave solution for (24). It will now be shown how this solution which we donote by g is obtained.

From the commutation rules (12) for α_i, β one has

$$[E + H_0][E - H_0] = [E - H_0][E + H_0] = (E^2 - m^2 + \nabla^2) = (| k |^2 + \nabla^2). \tag{25}$$

Consider now a solution of the differential equation

$$(E^2 - m^2 + \nabla_x^2)s(x, x') \equiv (| k |^2 + \nabla_x^2)s(x, x')$$
$$= \delta(x - x'). \tag{26}$$

(The subscript x on the operator ∇^2 indicates that the differentiations are to be carried out on the variable x rather than x'.)

Any solution $s(x,x')$ of equation (26) can be used to form a solution of equation (24). From (25) and (26) one has

$$[E - H_0][E + H_0]s(x, x')\ \delta(\gamma, \gamma') = \delta(x - x')\ \delta(\gamma, \gamma'). \tag{27}$$

Hence a solution $g(x, \gamma; x', \gamma')$ of (24) is

$$g(x, \gamma; x', \gamma') = [E + H_0]s(x, x')\ \delta(\gamma, \gamma'). \tag{28}$$

This method of obtaining influence functions for the Dirac operator is a well-known one. A solution $s_r(x,x')$ of (26) which leads to an outgoing wave is

$$s_r = -\frac{e^{i|k||x-x'|}}{4\pi\ | x - x' |}. \tag{29}$$

The influence function g_r obtained using s_r is in explicit form

$$g_r(x, \gamma; x', \gamma') = -\left\{\frac{-[| k |\ | x - x' | + i]}{| x - x' |^2} \sum_{i=1}^{3} \alpha_i(\gamma, \gamma')(x_i - x_i') - m\beta(\gamma, \gamma') + E\ \delta(\gamma, \gamma')\right\} \frac{e^{i|k||x-x'|}}{4\pi\ | x - x' |}. \tag{30}$$

Here $\alpha_i(\gamma, \gamma'), \beta(\gamma, \gamma')$ are the matrices which represent the operators α, β. The function g_r represents an outgoing wave because the time factor is e^{-iEt} (see (13)).

For large values of $| x |$, the expression for χ_{sc} becomes

$$\chi_{sc}(x, \gamma; E) = -\frac{e^{i|k||x|}}{4\pi\ | x |} \sum_{\gamma'} \left\{\left[E\ \delta(\gamma, \gamma') - | k | \sum_{i=1}^{3} \alpha_i(\gamma, \gamma')\eta_{1i} - m\beta(\gamma, \gamma')\right] \cdot \int e^{-i|k|(x', \eta_1)} q\chi(x', \gamma'; E)\, dx'\right\}. \tag{31}$$

In the above expressions and in those following the result of operating with an operator such as q on a function $f(x, \gamma)$ will be a function of x, γ which will be denoted by $qf(x, \gamma)$.

Here η_1 is a unit vector with components η_{1i} defined by

$$x = |x| \eta_1 \tag{32}$$

We should like to re-write (31) so that the amplitudes of the spherical waves are inner products in order that we may ultimately use the variational principle to obtain these amplitudes. This line of thought motivates our use of the identity

$$\left[E\ \delta(\gamma, \gamma') - |k| \sum_{i=1}^{3} \alpha_i(\gamma, \gamma')\eta_i - m\beta(\gamma, \gamma')\right]$$
$$= 2E \sum_{\tau} \chi(\gamma; E, \eta, \tau)\chi(\gamma'; E, \eta, \tau)^* \tag{33}$$

Though this identity is fundamental in our treatment, we shall not prove it, since it follows directly from (19) and (21).

If the incident wave part of the function $\chi(x, \gamma; E)$ has the direction η' and the value of τ is τ', we shall denote $\chi(x, \gamma; E)$ by $\chi(x, \gamma; E, \eta', \tau')$. We proceed to define the "spinor spherical wave" $\theta(x, \gamma; E, \eta, \tau)$ by

$$\theta(x, \gamma; E, \eta, \tau) = -\frac{Ee^{i|k||x|}}{2\pi\ |x|}\chi(\gamma; E, \eta, \tau). \tag{34}$$

The spinor spherical wave is analogous to the spinor plane wave (17). Furthermore, we define $T(E; \eta, \tau; \eta', \tau')$ by

$$T(E; \eta, \tau; \eta', \tau') = \sum_{\gamma'} \int \chi(\gamma'; E, \eta, \tau)^* e^{-i|k|(x', \eta)} q\chi(x', \gamma'; E, \eta', \tau')\, dx'$$
$$= \sum_{\gamma'} \int [q^*\chi(\gamma'; E, \eta, \tau)^*] e^{-i|k|(x', \eta)} \chi(x', \gamma'; E, \eta', \tau')\, dx'. \tag{35}$$

The expression for χ_{sc} takes the form for large $|x|$,

$$\chi_{sc}(x, \gamma; E) = \sum_{\tau} \theta(x, \gamma; E, \eta_1, \tau) T(E; \eta_1, \tau; \eta', \tau'). \tag{36}$$

The scattered wave may be regarded as the sum of two spinor spherical waves, each being characterized by a different value of τ. The function $T(E; \eta_1, \tau_1; \eta_2, \tau_2)$ may be regarded as the amplitude of the spinor spherical wave in the direction η_1 and with $\tau = \tau_1$ when the incident spinor plane wave has as its direction of propagation η_2 and its value of τ is τ_2. We have therefore obtained χ_{sc} in a form where $T(E; \eta, \tau; \eta', \tau')$ is an inner product to which, as will now be shown, a variational principle can be applied.

In order to show how the variational principle discussed abstractly earlier may be used to find the amplitudes $T(E; \eta, \tau; \eta', \tau')$ we need only show what quantities are to be identified with the quantities appearing in equation (3).

Let us first consider $T(E; \eta, \tau; \eta', \tau')$ as being the symmetric inner product of two vectors. We identify $\chi(x, \gamma; E, \eta', \tau')$ with the vector y of equation, (3). We see that the vector a' is to be identified with $q^*\ \chi(\gamma; E, \eta, \tau)^*\ e^{-i|k|}\ (x, \eta)$. From (15) and (23) we construct the equation corresponding to the first of equations (3).

$$q\chi(\gamma; E, \eta', \tau')e^{i|k|(x, \eta')} = q\chi(x, \gamma; E, \eta', \tau')$$
$$- q \sum_{\gamma'} \int g_r(x, \gamma; x', \gamma') q\chi(x', \gamma'; E, \eta', \tau')\, dx'. \tag{37}$$

The integral equation corresponding to the second equation (3) will also be given, since it is not difficult to obtain symmetric adjoint operator K'. Accordingly, the integral equation for Λ which is the function corresponding to the vector y' is

$$q^*\chi'(\gamma; E, \eta, \tau)^* e^{-i|k|(x,\eta)} = q^*\Lambda(x; \gamma; E, \eta, \tau)$$

$$- q^* \sum_{\gamma'} \int g_s(x, \gamma; x', \gamma') q^* \Lambda(x', \gamma', E, \eta, \tau)\, dx' \qquad (38)$$

where

$$g_s(x, \gamma; x', \gamma') = g_r(x', \gamma'; x, \gamma). \qquad (39)$$

Just as the integral equation (37) for $\chi(x, \gamma; E, \eta, \tau)$ was derived from the differential equation

$$H\chi(x, \gamma; E, \eta, \tau) = E\chi(x, \gamma; E, \eta, \tau) \qquad (40)$$

together with appropriate boundary conditions, the integral equation (38) for $\Lambda(x, \gamma; E; \eta, \tau)$ can be derived from a differential equation with certain boundary conditions. The differential equation satisfied by Λ is

$$H^*\Lambda(x, \gamma; E, \eta, \tau) = E\Lambda(x, \gamma; E, \eta, \tau) \qquad (41)$$

so that Λ is an eigenfunction of H^*. The boundary condition on Λ is that it is to be expressed as the sum of an incident wave Λ_{in} and a scattered wave Λ_{sc} such that Λ_{in} satisfies

$$H_0^*\Lambda_{in}(x, \gamma; E, \eta, \tau) = E\Lambda_{in}(x, \gamma; E, \eta, \tau) \qquad (42)$$

and Λ_{sc} behaves like the spherical wave $(e^{ik} \mid x \mid)/ \mid x \mid$ for large values of $\mid x \mid$.

We take as a suitable solution Λ_{in} of (42) the function χ_{in}^* where χ_{in} is given by (17.) The function Λ_{sc} as can be shown from (40), satisfies the equation

$$\Lambda_{sc}(x, \gamma; E) = \sum_{\gamma'} \int g_s(x, \gamma; x', \gamma') q^* \Lambda(x', \gamma'; E)\, dx' \qquad (43)$$

where the inverse operator $[E - H_0^*]^{-1}$ is represented by an integral operator with the kernel g_s.

The functional corresponding to $\lambda(v',v)$ whose extremal value is $1/[T(E; \eta, \tau; \eta', \tau')]$ is written as $1/T\{v',v\}$ and is given by

$$\frac{1}{T}\{v', v\}$$

$$= \frac{\sum_{\gamma} \int v'(x, \gamma)\{qv(x, \gamma) - q \sum_{\gamma'} \int g_r(x, \gamma; x', \gamma') qv(x', \gamma')\, dx'\}\, dx}{\left(\sum_{\gamma} \int v'(x, \gamma) q\chi(\gamma; E, \eta', \tau') e^{i|k|(x,\eta')}\, dx\right)\left(\sum_{\gamma} \int q^*\chi(\gamma; E, \eta, \tau)^* e^{-i|k|(x,\eta)} v(x, \gamma)\, dx\right)} \qquad (44)$$

where v' and v are the trial functions which approximate $\Lambda(x, \gamma; E, \eta, \tau)$ and $\chi(x, \gamma; E, \eta', \tau')$ respectively.

Having considered the case where $T(E; \eta, \tau; \eta', \tau')$ is a symmetric inner product, we shall now discuss the case in which this amplitude is considered a Hermitian inner product

of the two vectors $q\ \chi(\gamma; E, \eta, \tau)\ e^{i}\ |\ k\ |\ (\eta, x)$ and $\chi(x, \gamma; E, \eta', \tau')$. The identification of the vectors a and y and of the operator K are as before in the case of the symmetric inner product. The vector a' is now identified with $q\ \chi(\gamma;\ E,\ \eta,\ \tau)e^{i|k|(\eta,x)}$. The vector y' is identified with the function $\Omega(x, \gamma; E, \eta, \tau)$ which satisfies the following integral equation which corresponds to the second of equations (3).

$$q\chi(\gamma'; E, \eta, \tau)e^{i|k|(\eta,x)} = q\Omega(x, \gamma; E, \eta, \tau) - q \sum_{\gamma'} \int g_i(x, \gamma; x', \gamma')q\Omega(x', \gamma'; E, \eta, \tau)\, dx' \tag{45}$$

where

$$g_i(x, \gamma; x', \gamma') = g_r(x', \gamma'; x, \gamma)^* = -\left\{\frac{[|\ k\ |\ |\ x - x'\ | - i]}{|\ x - x'\ |^2} \sum_{i=1}^{3} \alpha_i(\gamma, \gamma')(x_i - x_i') - m\beta(\gamma, \gamma') + E\ \delta(\gamma, \gamma')\right\} \frac{e^{-i|k||x-x'|}}{4\pi\ |\ x - x'\ |}. \tag{46}$$

It can be shown that $\Omega(x, \gamma; E, \eta, \tau)$ as a solution of a differential equation with suitable boundary conditions. The function $\Omega(x, \gamma; E, \eta, \tau)$ satisfies the differential equation

$$H\Omega(x, \gamma; E, \eta, \tau) = E\Omega(x, \gamma; E, \eta, \tau) \tag{47}$$

so that Ω, like χ, is an eigenfunction of H. However, the function Ω has different boundary conditions than χ. The boundary condition on Ω is that it should be expressed as the sum of an incident part Ω_{in} and a "concentrating" part Ω_{con}. The function Ω_{in} satisfies

$$H_0\Omega_{in}(x, \gamma; E, \eta, \tau) = E\Omega_{in}(x, \gamma; E, \eta, \tau) \tag{48}$$

and is taken to be χ_{in} as given by (17). The function Ω_{con} is specified by the condition that Ω_{con} is to behave like an *inwardly* moving spherical wave for large values of $|\ x\ |$. Using the boundary condition that Ω_{con} represent an inward moving spherical wave for large values of $|\ x\ |$, the inverse operator $[E - H_0]^{-1}$ can be expressed as an integral operator while the function g_i is the solution of equation (24) expressed in the form (28) when the solution s of (26) is taken to be

$$s_i = -\frac{e^{-i|k||x-x'|}}{4\pi\ |\ x - x'\ |}$$

instead of s_r given by (29).

In the case of the Hermitian inner product the functional $1/T(v',v)$ is given by

$$\frac{1}{T}(v', v) = \frac{\sum_{\gamma} \int v'(x, \gamma)^* \{qv(x, \gamma) - q \sum_{\gamma'} \int g_r(x, \gamma; x', \gamma')qv(x', \gamma')\, dx'\}\, dx}{\left(\sum_{\gamma} \int v'(x, \gamma)^* q\chi(\gamma; E, \eta', \tau')e^{i|k|(x,\eta')}\, dx\right)\left(\sum_{\gamma} \int q^*\chi(\gamma; E, \eta, \tau)^* e^{-i|k|(x,\eta)}v(x, \gamma)\, dx\right)} \tag{49}$$

where v' and v approximate

$$\Omega(x, \gamma; E, \eta, \tau) \quad \text{and} \quad \chi(x, \gamma; E, \eta', \tau'),$$

respectively.

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A FORM OF NEWTON'S METHOD WITH CUBIC CONVERGENCE*

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For obtaining an approximation to a root of a transcendental equation $f(x) = 0$ Newton's method may well be unsatisfactory because it is only quadratically convergent, thus requiring considerable interpolation if the functions involved are scantily tabulated. On the other hand, the formulas of Stewart [1], Hamilton [2], Bodewig [3], and others, which offer cubic or higher convergence have the serious drawback of requiring the evaluation of second or higher derivatives. Formula (4) below, based on the generalized Taylor expansion of Hummel and Seebeck [4], offers cubic convergence in terms of $f(x)$ and $f'(x)$ evaluated at points on each side of the root.

Taking $n = m$ in the Hummel-Seebeck expansion,

$$f(x) = f(a) + (x - a)\frac{f'(a) + f'(x)}{2} + (x - a)^2\frac{f''(a) - f''(x)}{12} + \cdots, \tag{1}$$

we obtain two approximations to a root,

$$x - a = \frac{-2f(a)}{f'(a) + f'(x)} \quad \text{and} \quad x - b = \frac{-2f(b)}{f'(b) + f'(x)}. \tag{2}$$

We choose a and b so $f(a)$ and $f(b)$ are opposite in sign, $f'(a)$ and $f'(b)$ same sign. Elimination of $f'(x)$ in equations (2) yields

$$\frac{f(b)}{x - b} - \frac{f(a)}{x - a} + \frac{f'(b) - f'(a)}{2} = 0 \tag{3}$$

or, by an obvious procedure,

$$x = \frac{b + a}{2} - \frac{f(b) - f(a)}{f'(b) - f'(a)} \pm \left\{\left[\frac{b - a}{2} + \frac{f(b) - f(a)}{f'(b) - f'(a)}\right]^2 - \frac{2(b - a)f(b)}{f'(b) - f'(a)}\right\}^{1/2}, \tag{4}$$

where choice of the ambiguous sign is quite obvious.

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Proof of cubic convergence of the method closely follows the more general discussion of Bodewig. We take A as the true value of the first order root,

$$f(x) = (x - A)g(x), \qquad g(A) \neq 0, \tag{5}$$

and expand $g(x)$ in a generalized Taylor series, powers of $(x - A)$. Setting $x = a_n$, b_n in the series of equations (2) and carrying out the indicated division we obtain

$$\begin{aligned} \frac{x_{n+1} - a_n}{a_n - A} &= -\left[1 - (x_{n+1} - A)\,\frac{g(A) + g(x_{n+1})}{2g(A)} + P_1(x - A) + \cdots\right], \\ \frac{x_{n+1} - b_n}{b_n - A} &= -\left[1 - (x_{n+1} - A)\,\frac{g(A) + g(x_{n+1})}{2g(A)} + P_2(x - A) + \cdots\right], \end{aligned} \tag{6}$$

or, finally,

$$x_{n+1} - A = \frac{(a_n - A)(b_n - A)}{b_n - a_n}\,P(x - A), \tag{7}$$

where $P(x - A)$ represents a power series in $(a_n - A)$, $(b_n - A)$, $(x_{n+1} - A)$, quadratic terms and higher.

As a numerical example consider the first root of $x \tan x - 1 = 0$. Taking $a = 0.8$, $b = 0.9$ equation (4) yields the tabulated value 0.8603. Two or more applications of Newton's method will involve interpolation if one has only a two place table at hand. Estimates of the magnitude of error involved in interpolation by means of (4) have been found by Hummel and Seebeck [5].

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THE SECOND FUNDAMENTAL THEOREM OF ELECTRICAL NETWORKS*

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1. Introduction. This paper will deal with an extension of the work of W. H. Ingram and C. M. Cramlet[1] as discussed by J. L. Synge.[2] In addition it will be shown how their theories fit into a unified theory. The terminology of Synge's paper will be used.

A network may be represented by its Thévenin representation, i.e. by regarding its branches as consisting of impedances in series with constant voltage sources; or, it may be represented by its Norton representation, i.e. by regarding the branches as

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consisting of a constant current generator in parallel with an admittance. In the first representation the Kirchhoff nodal constraints become homogeneous in the branch currents and permit the introduction of mesh currents and the mesh equations, whereas in the second, the Kirchhoff mesh constraints become homogeneous in the branch voltages and allow the introduction of what P. Le Corbeiller[3] calls the basic voltages and the nodal equations. The theorem of Ingram and Cramlet which Synge describes as the fundamental theorem of electrical network theory is the fundamental theorem for the Thévenin representation of a network; the dual Norton theorem which is fully on a par with this theorem will be proved independently in this paper, and the relations of the two theorems will be discussed. In addition the nodal equations of G. Kron as given by P. Le Corbeiller will be deduced by a method analogous to that of Synge's. It may be noted that the nodal theorem is much easier to prove than the mesh theorem.

In section (2) we will prove the fundamental theorem for nodal networks and in section (3) we deduce Kron's equations. In section (4) we prove that the nodal equations have a unique solution; and in section (5) we show the relation of the theorems for the Thévenin and Norton representations. In section (2), (3) and (4) we shall consider only the Norton representation of a network.

2. Basic node voltages for connected networks. Let N be the number of nodes: $J_1, J_2, \cdots J_N$; and let B be the number of branches. In addition we assign a direction to each branch and define E_k as the voltage rise from the initial point to the terminal point of the k^{th} branch. A *path* on the network is defined as a finite sequence of branches such that each branch has one node in common with the preceding branch and the other node in common with the succeeding branch. The first branch is required to have only one node in common with the second and the last branch is required to have only one node in common with its predecessor. For two nodes J_p and J_q and a path directed from J_p to J_q we define V_{pq} as the sum of the voltage rises of those branches of the path which have the same sense as the path minus the voltage drops of those branches of the path which have a sense opposite to the sense of the path. We note that for the given path $V_{pq} = -V_{qp}$.

If we consider two paths joining J_p and J_q, these two paths taken in opposite senses form a closed path or mesh. Since there are no voltage sources in the network the sum of the voltage rises around the closed path is zero and thus V_{pq} for one path plus V_{qp} for the other path is zero. It follows from this that V_{pq} is independent of the path. If we select one node say J_1 as a ground node and define $V_p = V_{1p}$ $(p = 2, 3, \cdots n)$ then V_p, the potential relative to the ground, is also independent of the path. We shall call $V_2, V_3, \cdots, V_N$ the *basic voltages*. Since the voltage rise across a branch from J_p to J_q is $V_q - V_p$ we have:

Theorem I(a) If the branch voltages of a network satisfy Kirchhoff's mesh law then there are $N - 1$ basic voltages such that the branch voltages can be expressed in terms of them. These basic voltages are the potentials of the nodes relative to the ground node.

In addition if we prescribe $N - 1$ arbitrary basic voltages and define the voltage drops across the branches as above then we have:

Therorem I(b) An arbitrary set of $N - 1$ basic voltages for a network generates a unique set of branch voltage rises which satisfy Kirchhoff's mesh law.

If E is the column matrix of branch voltage rises $E_1, E_2, \cdots E_B$ then

$$E_p = \sum_{q=2}^{N} A_{pq} V_q \qquad (p = 1, 2, \cdots B) \tag{2.1}$$

where A_{pq} is defined as follows:

$$A_{pq} = \begin{cases} 1 \text{ if } J_q \text{ is the terminal node of the } p^{\text{th}} \text{ branch} \\ -1 \text{ if } J_q \text{ is the initial node of the } p^{\text{th}} \text{ branch} \\ 0 \text{ if } J_q \text{ is not on the } p^{\text{th}} \text{ branch.} \end{cases} \tag{2.2}$$

In matrix notation eq. (2.1) can be written

$$E = AV \tag{2.3}$$

Theorems $I(a)$ and $I(b)$ together may now be stated as:

Theorem II: For any connected network taking any node as the ground node there is a matrix A as in (2.2) such that:

(i) For arbitrary values of the associated basic voltages the branch voltages given by $E = AV$ satisfy Kirchhoff's mesh law on every mesh

(ii) For any set of branch voltages which satisfy Kirchhoff's mesh law on every mesh there is a set of basic voltages V such that

$$E = AV.$$

This theorem is the dual analogue of the "central theorem" of Synge, and can be generalized to apply to linearly independent combinations of nodes. In addition the restriction to networks which are not connected is readily eliminated by the use of ground nodes in each subnetwork.

3. Node equations. If we designate the current source of the r^{th} branch by I_r we can define B branch currents $U_1, U_2 \cdots U_B$ each of which consists of the corresponding branch source current and the branch current due to the branch potentials. In matrix notation

$$U = I - YE \tag{3.1}$$

where Y is the admittance matrix of the network whose elements are the self and mutual admittances of the branches.

We write Kirchhoff's node law as

$$\sum_{p=1}^{B} F_{qp} U_p = 0 \qquad (q = 2, 3, \cdots N) \tag{3.2}$$

where

$$F_{qp} = \begin{cases} 1 \text{ if } J_q \text{ is the terminal node of branch } p \\ -1 \text{ if } J_q \text{ is the initial node of branch } p \\ 0 \text{ if } J_q \text{ is not on branch } p. \end{cases} \tag{3.3}$$

In matrix form this becomes

$$FU = 0. \tag{3.4}$$

If we compare (2.2) and (3.3) we see that $F = A_t$ where A_t is the transpose of A. From

(3.1) we have

$$A_t(I - YE) = 0. \tag{3.5}$$

If we define I', the column matrix of basic currents by

$$I' = A_t I, \tag{3.6}$$

we can write (3.5) as

$$I' = A_t YE,$$

or by (2.3)

$$I' = A_t YAV. \tag{3.7}$$

Further, if we define Y' the basic node admittance matrix by

$$Y' = A_t YA, \tag{3.8}$$

then (3.7) can be written as

$$I' = Y'V. \tag{3.9}$$

Equations (2.3), (3.6), (3.8) and (3.9) are the basic equations of Kron's nodal method together with (4.1) below.

4. The existence and uniqueness of the solution of the nodal equations. We show here that if $I = 0$ implies $E = 0$ then the matrix Y' is non-singular, and hence the system of equations (3.9) has a unique solution. If $I = 0$ then by (3.6) $I' = 0$ and (3.9) becomes

$$Y' V = 0. \tag{4.1}$$

Since $V = 0$ is a solution of (4.1) our assertion will be proved if we show that it is the only solution. But if $I = 0$ implies that $E = 0$ then we must have $V = 0$. This is a consequence of the remark that if $E = 0$, the basic voltages of the nodes on any path are all equal, and since the network is connected and at least one node is connected to the ground node, the basic voltages of all the nodes are zero. Therefore Y' is non-singular and from (3.9) we get

$$V = (Y')^{-1} I',$$

and by (2.1)

$$E = A(A_t YA)^{-1} A_t I. \tag{4.2}$$

This is the fundamental formula for the solution of nodal networks by Kron's method.

5. Relation of the nodal theorem to the mesh theorem. For the given network let us consider any tree of the network. The voltage drops of the branches of the tree can be prescribed arbitrarily since there are no closed circuits on the tree to which Kirchhoff's mesh law applies. If we choose any node as J_1, construct V as above (see sec. 2.1), define the voltage drop across any two nodes as the difference between the values of V at the initial and terminal points of the node pair, and choose any sequence of nodes as defining a path, then Kirchhoff's mesh law will be satisfied for any closed path. Thus if we restore the branches, called chord branches, which were deleted to form the tree, then Kirchhoff's mesh law is certainly satisfied for meshes consisting of branch paths. Also if we replace these branches by their Thévenin equivalents then the currents in these branches which, as Synge has shown, are a complete set of mesh currents, are

determined by Ohm's law. If we now replace the tree branches of the network by their Thévenin equivalents then by Synge's theorem all the branch currents are determined and hence all the branch voltage drops are determined. Conversely if we assign the chord branch currents in the network regarded as consisting of Thévenin branches, then by Synge's theorem, the branch currents and hence the branch voltages are determined. This implies that the branch voltage drops of the original network are determined. This completes the proof of the equivalence of the nodal and mesh methods.

In addition, if the Thévenin equivalent branches are given then in Synge's notation

$$W = e - Zi \text{ or } e = W + Zi \tag{5.1}$$

where e is the branch voltage-source matrix, Z the branch impedance matrix and i is the matrix of branch currents. In the notation of this paper if the Norton equivalent branches are given then by

$$U = I - YE. \tag{5.2}$$

Since we are dealing with two representations of the same network we may identify i and U, and W and E. If Z is non-singular (5.1) can be written

$$i = Z^{-1}e - Z^{-1}E,$$

and by comparison with (5.2) we have

$$I = Z^{-1}e, \; YE = Z^{-1}E. \tag{5.3}$$

If we multiply (5.1) by C_t on the left and note that $e' = C_t e$ and $i = Ci'$ then we have

$$e' = C_t E + C_t Z C i'.$$

Since $e' = C_t Z C i'$ it follows that

$$C_t E = 0.$$

But by eq. (2.3) this may be written

$$C_t A V = 0.$$

Since V can be prescribed arbitrarily we have

$$C_t A = 0. \tag{5.4}$$

This last relation can be derived on purely topological grounds[4]. Also since in (5.3) E is arbitrary we have

$$Y = Z^{-1}. \tag{5.5}$$

Equations (5.3), (5.4) and (5.5) show the relations of the two methods.

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NOTE ON THE ELASTIC DISTORTION OF A CYLINDRICAL HOLE BY TANGENTIAL TRACTIONS ON THE INNER BOUNDARY.*

By Sisir Chandra Das (*Chandernagare College, India*)

Introduction. Elastic distortion of a cylindrical hole by localized hydrostatic pressure has been discussed by H. M. Westergaard[1] and C. J. Tranter.[2]

In this paper a few problems of elastic distortion of a cylindrical hole by tangential traction on its inner boundary are discussed. The first case considered is concerned with the problem of an infinite elastic plate having finite thickness with a cylindrical hole acted on by localized tangential traction on the inner boundary, the two faces of the plate being free. In the second case one face of the plate is supposed to be fixed and the other free, while the hole is acted on by uniform tangential traction throughout. In the last case a localized tangential traction is supposed to act over a narrow band on the inner boundary of an infinitely long cylindrical hole in an infinite elastic solid.

The solutions in the first two cases are obtained in terms of infinite series while in the last case it is expressed as an infinite integral.

1. Method of solution. We take the axis of the cylindrical hole as the axis of z. Using cylindrical co-ordinates and assuming $u = w = 0$ and v to be independent of θ, we have

$$\begin{aligned} e &= 0, \qquad 2w_r = -\frac{\partial v}{\partial z} \\ 2w_\theta &= 0, \qquad 2w_z = \frac{1}{r}\frac{\partial}{\partial r}(rv) \end{aligned} \tag{1.1}$$

and

$$\sigma_r = \sigma_\theta = \sigma_z = \tau_{rz} = 0,$$

$$\tau_{\theta z} = G\frac{\partial v}{\partial_z}, \qquad \tau_{r\theta} = G\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right) = Gr\frac{\partial}{\partial r}\left(\frac{v}{r}\right) \tag{1.2}$$

Two equations of equilibrium are identically satisfied and the third takes the form

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2} = 0 \tag{1.3}$$

One particular solution suitable for the problem is

$$v = \frac{A_0}{r}. \tag{1.4}$$

Also substituting

$$v = V\cos kz \qquad \text{or} \qquad V\sin kz \tag{1.5}$$

we get

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} - \left(k^2 + \frac{1}{r^2}\right)V = 0 \tag{1.6}$$

where V is a function of r only.

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[1] H. M. Westergaard, Kármán Anniversary Volume, 1941, p. 154

[2] C. J. Tranter, Quart. of Appl. Math., 4, 298 (1946).

As a solution to the above we take

$$V = c_k K_1(kr) \tag{1.7}$$

where K_1 is the modified Bessel function of the second kind of degree one.

We now consider the following cases.

2. The cylindrical hole in a large plate of finite thickness with a distribution of tangential traction over a narrow band. We take the plate to be of thickness $2L$ and the surfaces defined by $z = \pm L$. Both ends of the plate are free and the tangential traction is localized within the zone $z = \pm h$.

The boundary conditions to be satisfied are

1) $$\tau_{\theta z} = G\frac{\partial v}{\partial z} = 0, \quad \text{when} \quad z = \pm L \tag{2.1}$$

and 2) $$\begin{aligned} \tau_{r\theta} &= -S_1, && \text{when} \quad |z| < h \\ &= 0, && \text{when} \quad |z| > h \end{aligned} \tag{2.2}$$

Assume

$$v = \frac{A_0}{r} + \sum_{k=1}^{\infty} c_k K_1(kr) \cos kz \tag{2.3}$$

Then from (1.2) we have

$$\tau_{r\theta} = -\frac{2GA_0}{r^2} - \sum_{k=1}^{\infty} c_k k G K_2(kr) \cos kz \tag{2.4}$$

where K_2 is modified Bessel function of the second kind of the degree two.

From the condition (2.1) we get

$$k = \frac{n\pi}{L} \tag{2.5}$$

where n is any integer.

Also the conditions given by (2.2) will be satisfied if

$$A_0 = \frac{a^2}{2G} \cdot \frac{S_1 h}{L}$$

and

$$C_k = \frac{2LS_1 \sin(n\pi h/L)}{n^2\pi^2 G K_2(n\pi a/L)} \tag{2.6}$$

Hence

$$v = \frac{S_1 a^2 h}{2GLr} + \sum_{n=1}^{\infty} \frac{2LS_1 \sin(n\pi h/L) K_1(n\pi r/L) \cos(n\pi z/L)}{n^2\pi^2 G K_2(n\pi a/L)} \tag{2.7}$$

which is evidently convergent as for large values of x

$$\frac{K_1(x)}{K_2(x)} \approx 1.$$

3. The cylindrical hole with uniform tangential traction in a large plate one of whose faces is fixed and the other free. Here we take the plate to be of thickness L. The free surface is given by $z = L$, the face given by $z = 0$, being fixed. The uniform tangential traction is supposed to act throughout the hole.

The boundary conditions to be satisfied are.

1) $$v = 0, \quad \text{when} \quad z = 0 \tag{3.1}$$

2) $$\tau_{\theta z} = G\frac{\partial v}{\partial z} = 0, \quad \text{when} \quad z = L \tag{3.2}$$

and 3) $$\tau_{r\theta} = -S, \quad \text{when} \quad 0 < z < L \tag{3.3}$$

at the cylindrical surface $r = a$.

Assuming

$$v = \sum_{k=1}^{\infty} D_k K_1(kr) \sin kz, \tag{3.4}$$

we get from (1.2)

$$\tau_{r\theta} = -\sum_{k=1}^{\infty} D_k k G K_2(kr) \sin kz. \tag{3.5}$$

We find that the condition (3.1) is evidently satisfied and condition (3.2) will be satisfied if

$$k = \frac{(2n + 1)\pi}{2L} \tag{3.6}$$

where n is any integer.

The condition (3.3) will be satisfied if

$$D_k = \frac{8SL}{(2n + 1)^2\pi^2 GK_2((2n + 1)\pi a/2L)} \tag{3.7}$$

Hence

$$v = \sum_{n=1}^{\infty} \frac{8SLK_1((2n + 1)\pi r/2L) \sin ((2n + 1)\pi a/2L)}{(2n + 1)^2\pi^2 GK_2((2n + 1)\pi a/2L)} \tag{3.8}$$

which is evidently convergent.

4. The cylindrical hole in an infinite solid under a tangential traction over a narrow band. We now consider the case of an infinite solid having an infinitely long cylindrical hole acted on by a tangential traction which is operating over a narrow band of breadth $2h$.

The boundary conditions here are

$$\begin{aligned} \tau_{r\theta} &= -S_1 \quad \text{when} \quad |z| < h \\ &= 0 \quad \text{when} \quad |z| > h \end{aligned} \tag{4.1}$$

at the surface of the cylinder $r = a$.

The conditions (4.1) can be expressed in the form

$$(\tau_{r\theta})_{r=a} = \frac{-2S_1}{\pi} \int_0^\infty \frac{\sin ht \cos zt}{t} \, dt \tag{4.2}$$

We assume

$$v = \int_0^\infty c(t) K_1(tr) \cos zt \, dt \tag{4.3}$$

as a solution of the equation of equilibrium (1.3) where $c(t)$ is a function of t only.

The boundary condition (4.2) will be satisfied if

$$c(t) = \frac{2S_1 \sin ht}{t^2 \pi G K_2(ta)}. \tag{4.4}$$

Therefore

$$v = \int_0^\infty \frac{2S_1 \sin ht \cos zt K_1(tr) \, dt}{t^2 \pi G K_2(ta)}. \tag{4.5}$$

In conclusion I offer my sincere thanks to Dr. B. Sen for his help in the preparation of this paper.

ON CERTAIN SOLUTIONS OF A PENDULUM-TYPE EQUATION*

By GEORGE SEIFERT (*University of Nebraska*)

Introduction. In the study of the oscillations of a synchronous motor around its average angular velocity, a differential equation of the following type, the so-called pendulum-type arises [1]:

$$\frac{d^2\theta}{dt^2} + f(\theta) \frac{d\theta}{dt} = g(\theta) \tag{1}$$

where $f(\theta)$ and $g(\theta)$ are functions of period 2π in θ.

It has been shown [2] that in the case where $f(\theta) = \alpha > 0$, a constant, and $g(\theta) = \beta - \sin\theta$, where β is a constant such that $0 < \beta < 1$, there exists a constant $\alpha_c = \alpha_c(\beta) > 0$ such that if $\alpha < \alpha_c$, eq. (1) will have a solution $\theta(t)$ such that if $\theta'(t) = y(\theta)$, then $y(\theta) = y(\theta + 2\pi)$ for all t, while if $\alpha \geq \alpha_c$, no such solutions exist. Following Vlasov [3] and Minorsky [4], we call any such solution of eq. (1) a periodic solution of the second kind. Physically, such a solution corresponds to a subsynchronous level of performance of the motor described by eq. (1). It is known also [5] that questions of stability of solutions of eq. (1) with respect to the points of equilibrium of (1) involve questions of existence of such solutions.

The purpose of this note is to exhibit a set of explicit conditions on $f(\theta)$ and $g(\theta)$ which insure the existence of periodic solutions of the second kind for (1). Since it has already been noted [5] that if $f(\theta) > 0$ and either $g(\theta) < 0$ or $g(\theta) > 0$ for all θ, there will exist such solutions for eq. (1), we restrict ourselves to the case where $f(\theta) > 0$ and the equation $g(\theta) = 0$ has a finite number of roots in $0 \leq \theta < 2\pi$.

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We also show (see Remark 1) that in the special case where $f(\theta) = \alpha > 0$ and $g(\theta) = \beta - \sin\theta$ where $0 < \beta < 1$, these conditions lead to a somewhat larger lower bound on α_c that was obtained by Tricomi [2].

Theorem. Let $f(\theta)$ and $g(\theta)$ be twice-differentiable functions of period 2π in θ, $f(\theta) > 0$, and the roots $\{\theta_i\}$, $i = 0, 1, \cdots, 2n$, of the equation $g(\theta) = 0$ in the interval $\theta_0 \le \theta \le \theta_0 + 2\pi = \theta_{2n}$ be such that $g'(\theta_0) > 0$ and $g'(\theta_i) \ne 0$. Then eq. (1) will have a periodic solution of the second kind, provided each of the conditions below holds for each integer j such that $0 < j \le n$:

$$\text{(i)}\quad -f'(\theta)g'(\theta) < 2g''(\theta)\left(\frac{f^2(\theta)}{4} + g'(\theta)\right)^{1/2} \qquad \text{for} \quad \theta = \theta_{2j}\ ;$$

$$\text{(ii)}\quad f'(\theta)g(\theta) < 2(g'(\theta_{2j}) - g'(\theta))\left(\frac{f^2(\theta)}{4} + g'(\theta_{2j})\right)^{1/2} \qquad \text{for} \quad \theta_{2j-1} < \theta < \theta_{2j}\ ;$$

(iii) the equation $F_j(\theta) = f(\theta)h_j(\theta) - g(\theta) = 0$ has $m \ge 2$ roots $\{\rho_j^{(i)}\}$, $i = 1, 2, \cdots, m$, such that $\theta_{2j-2} < \rho_j^{(1)} < \rho_j^{(2)} < \cdots < \rho_j^{(m)} < \theta_{2j}$, for which $F'(\rho_j^{(i)}) \ne 0$ when $i = 1, 2$, and

$$\int_{\theta_j^*}^{\theta_{2j}} g(\theta)\, d\theta \ge \int_{\rho_j^{(2)}}^{\theta_{2j}} f(\theta)h_j(\theta)\, d\theta + h_j(\rho_j^{(2)}) \int_{\theta_j^*}^{\rho_j^{(1)}} f(\theta)\, d\theta \tag{2}$$

$$h_j(\theta) = \begin{cases} h_j^{(1)}(\theta) & \text{for} \quad \theta_{2j-1} < \theta \le \theta_{2j} \\ h_j^{(2)}(\theta) & \text{for} \quad \theta_{2j-2} \le \theta \le \theta_{2j-1}\ , \end{cases}$$

$$h_j^{(1)}(\theta) = \int_\theta^{\theta_{2j}} \left[\frac{f(s)}{2} + \left(\frac{f^2(s)}{4} + g'(\theta_{2j})\right)^{1/2}\right] ds,$$

$$h_j^{(2)}(\theta) = \int_\theta^{\theta_{2j-1}} f(s)\, ds + h_j^{(1)}(\theta_{2j-1}),$$

and θ_j^* is the largest root of $f(\theta)h_j(\rho_j^{(2)}) = g(\theta)$ such that $\theta_j^* \le \rho_j^{(1)}$

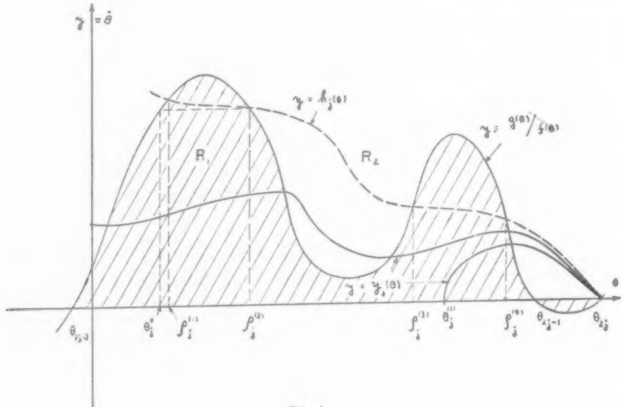

FIG. 1

Remark 1. In the case for which $g(\theta) = \beta - \sin\theta$, $0 < \beta < 1$, and $f(\theta) = \alpha > 0$, we put $\theta_1 = \arcsin\beta$, $0 < \theta_1 < \pi/2$, $\theta_2 = \pi - \theta_1$, and $\theta_0 = \theta_2 - 2\pi$. Conditions (i) and (ii) of the theorem are trivially satisfied. If we require that

$$\frac{\alpha}{2} + \left(\frac{\alpha^2}{4} + \cos\theta_1\right)^{1/2} \leq \frac{\beta}{\alpha\theta_2}, \tag{3}$$

then $F(\theta) = \alpha h_1(\theta) - \beta + \sin\theta = 0$ will have two roots, $\rho_1^{(1)} < \rho_1^{(2)}$ for which $0 \leq \rho_1^{(2)} < \theta_1$, and clearly $h_1(\rho_1^{(2)}) \leq \beta/\alpha$, which implies that θ_1^* is such that $\theta_0 < \theta_1^* \leq -\pi$. Hence inequality (2) of condition (iii) holds since

$$\beta(\theta_2 - \theta_1^*) + \cos\theta_2 - \cos\theta_1^* > \frac{\beta}{\alpha}\alpha(\theta_2 - \theta_1^*) \geq h_1(\rho_1^{(2)})\alpha(\theta_2 - \theta_1^*),$$

and thus inequality (3), which can be written as follows:

$$\alpha \leq \frac{\beta}{\theta_2}\left(\frac{\beta}{\theta_2} + \cos\theta_1\right)^{-1/2},$$

insures the existence of periodic solutions of the second kind in this case. The right hand member of this last inequality will, then, serve as a lower bound on α_c defined in the introduction. Tricomi's lower bound on α_c is

$$\frac{\beta}{\pi}\left(\frac{\beta}{\pi} + \cos\theta_1\right)^{-1/2} \quad [2].$$

Proof of the theorem. If we put $d\theta/dt = y$, eq. (1) becomes $dy/dt = g(\theta) - f(\theta)\, y$, and the equation of the phase trajectories becomes

$$\frac{dy}{d\theta} = \frac{g(\theta) - f(\theta)y}{y}. \tag{4}$$

The singularities (in the sense of Poincaré) of eq. (4) consist of the points $(\theta_i, 0)$ of the (θ, y) cartesian phase plane, the θ_i being the zeros of $g(\theta)$. An analysis of these singularities shows that the points $(\theta_{2i}, 0)$ $i = 0, 1, \cdots, n$, for which $g'(\theta_{2i}) > 0$, are saddle points, and the phase trajectories on these points, the so-called separatrices, have slopes given by

$$-\frac{f(\theta_{2i})}{2} \pm \left(\frac{f^2(\theta_{2i})}{4} + g'(\theta_{2i})\right)^{1/2}$$

respectively.

We note that in the region R_1 of the phase plane bounded by the curve of the equation $y = g(\theta)/f(\theta)$ and the θ-axis all phase trajectories have positive slopes, while in remaining part of the phase plane, R_2, the trajectories have non-positive slopes, having slope zero only at points on the curve of $y = g(\theta)/f(\theta)$.

For fixed j, let us consider the solution $y_j(\theta)$ of eq. (4) corresponding to the phase trajectory going into $(\theta_{2j}, 0)$ from the left with negative slope; i.e. $y_j(\theta)$ is such that

$$\lim_{\theta\to\theta_{2j}-} y_j'(\theta) = -\frac{f(\theta_{2j})}{2} - \left(\frac{f^2(\theta_{2j})}{4} + g'(\theta_{2j})\right)^{1/2}.$$

Clearly, we have either $y_j(\theta) > 0$ for $\theta_{2j-2} < \theta < \theta_{2j}$, or there exists a point $(\theta_j^{(1)}, 0)$ such that $\theta_{2j-2} < \theta_j^{(1)} < \theta_{2j}$ for which $y_j(\theta_j^{(1)}) = 0$. (See fig. 1). Suppose such a $\theta_j^{(1)}$ exists; then the solution $y_{j-1}^{(1)}(\theta)$ for which $y_{j-1}^{(1)}(\theta_{2j-2}) = 0$ and $\lim_{\theta\to\theta^+_{2j-2}} y'^{(1)}_{j-1}(\theta) > 0$

must be such that $y_{j-1}^{(1)}(\theta) > 0$ for $\theta_{2i-2} < \theta \leq \theta_{2i}$; for if not, its trajectory would have to intersect the θ axis in $\theta_{2j-1} \leq \theta \leq \theta_{2j}$, which would imply the intersection of the trajectories corresponding to $y_{j-1}^{(1)}(\theta)$ and $y_j(\theta)$, which is impossible. Hence $y_{j-1}^{(1)}(\theta) > 0$ for $\theta_{2j-2} < \theta \leq \theta_{2j}$.

If this holds for $j = 1, 2, \cdots, n$, it is easily seen that $y_0^{(1)}(\theta_{2n}) > 0$, for if not, its trajectory would have to intersect that of some one of the $y_j^{(1)}(\theta), j = 1, 2, \cdots, n$, which is impossible. Hence, $0 = y_0^{(1)}(\theta_0) < y_0^{(1)}(\theta_{2n}) = y_0^{(1)}(\theta_0 + 2\pi)$. On the other hand, the solution $y_0^{(2)}(\theta)$ for which $y_0^{(2)}(\theta_0) = M$, where M is such that $g(\theta)/f(\theta) < M$ for all θ, has the property that $y_0^{(2)}(\theta_0) > y_0^{(2)}(\theta_0 + 2\pi)$. By an argument used by Amerio [5], which we omit, we conclude that there exists for eq. (4) a solution such that $y(\theta) = y(\theta + 2\pi)$ for all θ.

We need only show, then, that the existence of some integer j, $0 < j \leq n$, such that $y_j(\theta) > 0$ for $\theta_{2j-2} < \theta < \theta_{2j}$ will contradict the conditions of the theorem. To this end, we show that $y_j(\theta) < h_j(\theta)$ for $\rho_j^{(2)} \leq \theta < \theta_{2j}$, and $y_j(\theta) < h_j(\rho_j^{(2)})$ for $\theta^* \leq \theta < \rho_j^{(2)}$ where $\rho_j^{(2)}$ and θ^* are defined in the statement of the theorem.

We first note that $\theta_{2j-2} < \rho_j^{(2)} < \theta_{2j-1}$. Define $\phi_j(\theta) = h_j(\theta) - y_j(\theta)$; we have, clearly $\phi_j'(\theta)_{2j}) = \phi_j(\theta_{2j}) = 0$ and will show that condition (i) implies $\phi_j''(\theta_{2j}) > 0$. We have from eq. (4),

$$y_j''(\theta_{2j}) = \lim_{\theta\to\theta_{2j}^-} \left\{\frac{-[y_j'(\theta)]^2 - f(\theta)y_j'(\theta) + g'(\theta)}{y_j(\theta)} - f'(\theta)\right\}$$

$$= \lim_{\theta\to\theta_{2j}^-} \left\{\frac{-2y_j'(\theta)y_j''(\theta) - f(\theta)y_j''(\theta) - f'(\theta)y_j'(\theta) + g''(\theta)}{y_j'(\theta)}\right\}$$

$$- f'(\theta_{2j})$$

by L'Hospital's rule. From this and the fact that $h_j'(\theta_{2j}) = y_j'(\theta_{2j})$, we obtain

$$y_j''(\theta_{2j}) = \frac{g''(\theta_{2j}) - 2f'(\theta_{2j})h_j(\theta_{2j})}{3h_j'(\theta_{2j}) + f(\theta_{2j})},$$

and by a routine calculation which we omit we note that the assumption $h_j''(\theta_{2j}) \leq y_j''(\theta_{2j})$ would contradict condition (i). Hence $\phi_j''(\theta_{2j}) > 0$ and there is an $\epsilon > 0$ such that for $\theta_{2j} - \epsilon < \theta < \theta_{2j}$, we have $\phi_j(\theta) > 0$; i.e. $h_j(\theta) > y_j(\theta)$.

We show next that for any θ such that $\theta_{2j-1} \leq \theta < \theta_{2j}$, the assumption $\phi_j(\theta) = 0$ would contradict condition (ii). For suppose for $\theta = \theta_a$ in this interval, $\phi_j(\theta_a) = 0$; then there exists a ξ, $\theta_a < \xi < \theta_{2j}$, such that $\phi_j'(\xi) = 0, \phi_j''(\xi) \leq 0$. From eq. (4) we have again

$$y_j''(\theta) = \frac{-[y_j'(\theta)]^2 - f(\theta)y_j'(\theta) + g'(\theta)}{y_j(\theta)} - f'(\theta), \tag{5}$$

and also

$$y_j(\theta) = \frac{g(\theta)}{y_j'(\theta) + f(\theta)};$$

hence eq. (5) becomes

$$y_j''(\theta) = \frac{\{-[y_j'(\theta)]^2 - f(\theta)y_j'(\theta) + g'(\theta)\}(y_j'(\theta) + f(\theta))}{g(\theta)} - f'(\theta).$$

If we put $\theta = \xi$ in this last equation, use $h_j'(\xi) = y_j'(\xi)$, and recall that $g(\xi) < 0$, a routine calculation will again show that the condition $h_j''(\xi) \leq y_j''(\xi)$ contradicts condition (ii).

We next consider the interval $\rho_j^{(2)} \leq \theta \leq \theta_{2j-1}$. Recall that in this interval $g(\theta) \geq 0$, the equality being possible only for $\theta = \theta_{2j-1}$. Note also that $y_j'(\theta) = -f(\theta)$ for $\theta = \theta_{2j-1}$. Now if for $\theta = \theta_b$, $\rho_j^{(2)} \leq \theta_b < \theta_{2j-1}$, we have $y_j(\theta_b) = h_j(\theta_b) = h_j^{(2)}(\theta_b)$, there exists an

η such that $\theta_b < \eta < \theta_{2i-1}$ for which $y_i'(\eta) < -f(\eta)$; but this clearly implies

$$y_i(\eta) = \frac{g(\eta)}{y_i'(\eta) + f(\eta)} < 0$$

which contradicts the assumption $y_i(\theta) > 0$ for $\theta_{2i-2} < \theta < \theta_{2i}$. We thus conclude that $y_i(\theta) < h_i(\theta)$ for $\rho_i^{(2)} \leq \theta \leq \theta_{2i-1}$.

Finally, we note that for $\theta_i^* \leq \theta \leq \rho_i^{(2)}$, the curve of $y = f(\theta)/g(\theta)$ is above the line $y = h_i^{(2)}(\rho_i^{(2)})$; this clearly implies that $y_i(\theta) < h_i(\rho_i^{(2)})$ for θ in this interval.

To sum up, we have that for $\rho_i^{(2)} \leq \theta < \theta_{2i}$ we have $y_i(\theta) < h_i(\theta)$, while for $\theta_i^* \leq \theta \leq \rho_i^{(2)}$ we have $y_i(\theta) < h_i(\rho_i^{(2)})$.

We now substitute $y = y_i(\theta)$ in eq. (4), multiply by $y_i(\theta)$, and integrate from θ_i^* to θ_{2i}; we obtain

$$-[y_i(\theta_i^*)]^2/2 = \int_{\theta_i^*}^{\theta_{2i}} g(\theta)\, d\theta - \int_{\theta_i^*}^{\theta_{2i}} f(\theta)y_i(\theta)\, d\theta.$$

Hence

$$\int_{\theta^*}^{\theta_{2i}} g(\theta)\, d\theta < \int_{\theta_i^*}^{\theta_{2i}} f(\theta)y_i(\theta)\, d\theta.$$

However, clearly,

$$\int_{\theta_i^*}^{\theta_{2i}} f(\theta)y_i(\theta)\, d\theta < \int_{\rho_i^{(2)}}^{\theta_{2i}} f(\theta)h_i(\theta)\, d\theta + h_i(\rho_i^{(2)}) \int_{\theta_i^*}^{\rho_i^{(2)}} f(\theta)\, d\theta$$

which, taken with the previous inequality, contradicts condition (iii). This proves the existence of $\theta_i^{(1)}$, such that $\theta_{2i-2} \leq \theta_i^{(1)} \leq \theta_{2i}$, for which $y_i(\theta_i^{(1)}) = 0$; hence, the theorem.

Remark 2. In the special case considered in Remark 1, we note that

$$h_1'(\theta) \not\leq \frac{\beta - \sin\theta - \alpha h_1(\theta)}{h_1(\theta)} \qquad \text{for} \qquad 0 < \theta < \theta_1 .$$

This shows that $y = h_1(\theta)$ is not a curve for which each trajectory crossing it from the right passes below it; if it were such a curve, conditions (i) and (ii) clearly would have been unnecessary in this case.

Remark 3. Since it is usually a question of imposing conditions on the parameters in $f(\theta)$, it is sometimes more convenient to replace inequality (2) of condition (iii) by the simpler but stronger condition:

$$\int_{\theta_i^*}^{\theta_{2i}} g(\theta)\, d\theta \geq h_i(\rho_i^{(2)}) \int_{\theta_i^*}^{\theta_{2i}} f(\theta)\, d\theta.$$

On the other hand, condition (iii) can clearly, at the expense of simplicity, be weakened. We omit the details.

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PULSED SURFACE HEATING OF A SEMI-INFINITE SOLID*

By J. C. JAEGER (*Australian National University*)

1. Introduction. In many practical systems heat is supplied to the surface of a solid in regular rectangular pulses, so that the flux of heat F at the surface at time t will be

$$\left.\begin{aligned} F &= F_0, \qquad nT < t < nT + T_1 \\ F &= 0, \qquad nT + T_1 < t < (n+1)T, \end{aligned}\right\} n = 0, 1, 2, \cdots \tag{1}$$

where F_0, T, and T_1 are constants and the solid is supposed to be at zero temperature when $t = 0$. Such problems occur, for example, in "on-off" heating, in heat generation by friction over portion of the surface of a rotating cylinder [1], in the rotating anode X-ray tube[2], in the heating of machine guns, in the heating of the anode of a magnetron, and so on.

It is usually desired to know the surface temperature after steady conditions have been attained, and, in particular, the temperature at the end of a heating interval. Fourier series are not very suitable for the treatment of such problems (since their convergence is very slow at the most interesting values of the time) and alternative methods have been given by Weber[3] and Oosterkamp[2], but the most powerful and widely applicable method seems to be that of the Laplace transformation[4] in a form which is essentially equivalent to the steady-state operational calculus of Waidelich[5].

To illustrate the method the most important case, that of the semi-infinite solid heated over the whole of its surface, is discussed in some detail in §2: the results of this section are frequently useful as approximations for the case of a finite solid similarly heated.

The more difficult problem of the semi-infinite solid with heat supplied over a circular area of its surface is of interest in connection with rotating anode X-ray tubes and is discussed in §3. Results for pulsed point and line sources in infinite medium are given in §4.

2. The semi-infinite solid heated over the whole of its surface.

The semi-infinite solid $x > 0$, of conductivity k and diffusivity α, initially at zero temperature, is heated over the whole of the surface $x = 0$ by the flux (1). *It is required to find its surface temperature for large values of the time.*

If v is the temperature in the solid we have to solve

$$\frac{\partial^2 v}{\partial x^2} - \frac{1}{\alpha}\frac{\partial v}{\partial t} = 0, \qquad x > 0, \qquad t > 0, \tag{2}$$

with

$$v = 0, \text{ when } t = 0, x > 0, \tag{3}$$

$$-k\frac{\partial v}{\partial x} = F, \qquad x = 0, \qquad t > 0, \tag{4}$$

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[1]Jaeger, Phil. Mag., (7), **35**, 169 (1944).

[2]Oosterkamp, Philips Res. Rep., **3**, 161 (1948).

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[4]Carslaw and Jaeger, *Operational Methods in Applied Mathematics*, Ed. 2 (1948), §129.

[5]Waidelich, Proc. Inst. Radio Engrs., 23, 78P (1946).

and v finite as $x \to \infty$, where F is given by (1). Writing

$$v^* = \int_0^\infty e^{-st} v \, dt \tag{5}$$

for the Laplace transform of v, with a similar notation for that of F, etc., we have from (1)

$$F^* = \frac{F_0(1 - e^{-sT_1})}{s(1 - e^{-sT})} \tag{6}$$

and, by the usual Laplace transformation procedure[4]

$$v^* = \frac{F_0\alpha^{1/2}(1 - e^{-sT_1})e^{-xs^{1/2}\alpha^{1/2}}}{ks^{3/2}(1 - e^{-sT})}. \tag{7}$$

This gives for the surface temperature v_0 at $x = 0$,

$$v_0 = \frac{2F_0}{k}\left(\frac{\alpha t}{\pi}\right)^{1/2}, \qquad 0 < t < T_1, \tag{8}$$

$$v_0 = \frac{2F_0\alpha^{1/2}}{k\pi^{1/2}}\{t^{1/2} - (t - T_1)^{1/2}\}, \qquad T_1 < t < T, \tag{9}$$

and so on. Also, the inversion theorem for the Laplace transformation gives

$$v_0 = \frac{F_0\alpha^{1/2}}{2\pi ikT}\int_{\gamma - i\infty}^{\gamma + i\infty}\left\{\frac{T - Te^{-sT_1} - T_1 + T_1e^{-sT}}{1 - e^{-sT}} + T_1\right\}e^{st}s^{-3/2}\,ds, \tag{10}$$

for all values of t, where γ is a positive constant. The integrand of (10) has a branch point at $s = 0$ and simple poles at

$$s = \pm 2n\pi i/T, \qquad n = 1, 2, \cdots. \tag{11}$$

When the integral (10) is evaluated by contour integration in the usual way, the residues at these poles give rise to a series of terms with period T, and the sum of this series, which we shall denote by v_P, is the required steady periodic term. The final result is

$$v_0 = \frac{2F_0T_1}{kT}\left(\frac{\alpha t}{\pi}\right)^{1/2} + v_P - \frac{2F_0\alpha^{1/2}}{\pi kT}\int_0^\infty \frac{e^{-u^2t}(T - Te^{u^2T_1} - T_1 + T_1e^{u^2T})\,du}{u^2(1 - e^{u^2T})}, \tag{12}$$

where the first term of (12) comes from the last term in the brace in (10), and the integral in (12) comes from the branch point in the first term in the brace in (10).

For large values of the time the integral in (12) is negligible, and v_0 is given by the steady periodic part v_P superposed on the first term of (12) which corresponds to steady heating by the average flux F_0T_1/T commencing at $t = 0$.

For small values of t the result (12) must agree with (8) for $0 < t < T_1$, and with (9) for $T_1 < t < T$, and this gives an integral expression for v_P over the whole of a period. Writing

$$a = T_1/T, \qquad Q = F_0T_1, \tag{13}$$

the value of v_P at time bT after the beginning of a period is

$$v_P = \frac{2Q}{ka}\left(\frac{\alpha}{\pi T}\right)^{1/2}\{(1-a)b^{1/2} - \pi^{-1/2}I(a,b)\}, \qquad 0 < b < a \tag{14}$$

$$v_P = \frac{2Q}{ka}\left(\frac{\alpha}{\pi T}\right)^{1/2}\{(1-a)b^{1/2} - (b-a)^{1/2} - \pi^{-1/2}I(a,b)\}, \qquad a < b < 1, \tag{15}$$

where

$$I(a,b) = \int_0^\infty \frac{e^{-b\xi^2}\{(1-a)e^{-\xi^2} - e^{-(1-a)\xi^2} + a\}\,d\xi}{\xi^2(1-e^{-\xi^2})}. \tag{16}$$

These integrals are easy to evaluate numerically. The values of a, the ratio of the heating time T_1 to the period T, vary greatly in practice. For on-off switching and frictional heating of a rotating cylinder they may be relatively large, but in other systems, such as machine guns or the anodes of X-ray tubes, they are usually of the order of 0.02 or less. Values of v_P at time of bT after the beginning of a period are shown in Curves I and II of Fig. 1 for $a = 0.5$ and 0.1, respectively, for the same quantity of heat Q emitted per unit area per cycle.

Curve III of Fig. 1 corresponds to the case $a \to 0$, that is, to the quantity of heat Q per unit area being supplied instantaneously at the beginning of each period so that (1) is replaced by

$$F = Q\sum_{n=0}^{\infty}\delta(t - nT), \tag{17}$$

where $\delta(t)$ is the Dirac delta function.

In this case (14) and (15) are replaced by

$$v_P = \frac{Q}{k}\left(\frac{\alpha}{\pi T}\right)^{1/2}\left\{b^{-1/2} - 2b^{1/2} - \frac{2}{\pi^{1/2}}\int_0^\infty \frac{e^{-b\xi^2}\{1-(\xi^2+1)e^{-\xi^2}\}\,d\xi}{\xi^2(1-e^{-\xi^2})}\right\}. \tag{18}$$

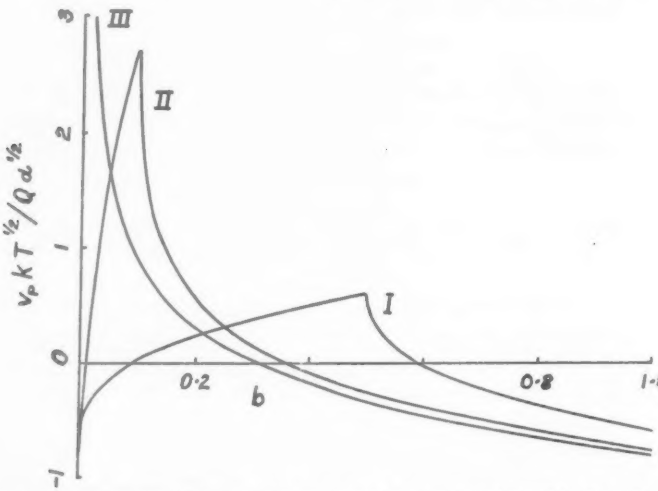

FIG. 1. Periodic temperature oscillations of the surface of a semi-infinite solid when the same quantity of heat is liberated over half the cycle (Curve I), over 1/10 of the cycle (Curve II), and instantaneously (Curve III).

The most important quantities in practice are the temperatures at the beginning and end of a heating interval, obtained by taking $b = 0$ and $b = a$, respectively in (14). The variation of these with the parameter $a = T_1/T$ for the same total quantity of heat Q supplied per unit area per cycle is shown in Fig. 2, Curves I and II.

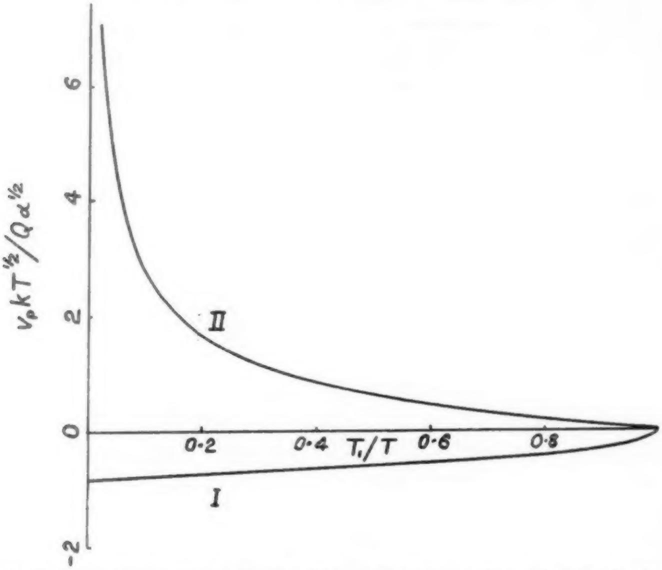

FIG. 2. Periodic Temperature at the beginning (Curve I) and end (Curve II) of a heating interval.

Finally it may be remarked that there is no difficulty in writing down expressions for the temperature at depth x. For example, for the surface flux (17) the temperature at depth x at a time t after the beginning of heating which is so long that the starting transient has disappeared is

$$\frac{2Q(\alpha t)^{1/2}}{kT} \operatorname{ierfc} \frac{x}{2(\alpha t)^{1/2}} + v_P , \tag{19}$$

where the value of the periodic part at time bT after the beginning of a heating interval is

$$v_P = \frac{Q\alpha^{1/2}}{k(\pi bT)^{1/2}} e^{-c^2/4b} - \frac{2Q\alpha^{1/2}b^{1/2}}{kT^{1/2}} \operatorname{ierfc} \frac{c}{2b^{1/2}} - \frac{2Q\alpha^{1/2}}{\pi k T^{1/2}} \int_0^\infty \frac{e^{-b\xi^2}[1 - (1 + \xi^2)e^{-\xi^2}] \cos c\xi}{\xi^2(1 - e^{-\xi^2})} d\xi, \tag{20}$$

where

$$\operatorname{ierfc} x = \int_x^\infty \operatorname{erfc} \xi \, d\xi,$$

$c = x(\alpha T)^{-\frac{1}{2}}$, and a and Q are defined in (13).

It follows from (19) that, for large values of the time t, the temperature is accurately the sum of the periodic oscillation v_P whose mean value is zero and a term corresponding to heating for time t by the constant flux Q/T, the mean value of the applied flux. The

same remark applies to all the other cases discussed. There seems to be no justification for the suggestion of Comenetz[6] that the temperature behaves on the average as if the constant flux Q/T had been supplied for time $t + \frac{1}{2}T$.

3. The semi-infinite solid heated over a circular area.

Suppose the semi-infinite solid is heated by flux (1) applied over a circle of radius R in its surface, there being no loss of heat from the remainder of the surface. When steady conditions have been attained, the temperature at the centre of the circle at time bT after the beginning of a period is found to be

$$\frac{RQ}{kT} + v_P , \tag{21}$$

where, now, the steady periodic part v_P is given by

$$v_P = \frac{Q\alpha^{1/2}}{kT^{1/2}} \left\{ \frac{2b^{1/2}}{a\pi^{1/2}} (1 - e^{-c^2/4b}) + \frac{c}{a} \operatorname{erfc} \frac{c}{2b^{1/2}} - c \right.$$
$$\left. + \frac{2}{\pi a} \int_0^\infty \frac{e^{-b\xi^2}\{e^{-(1-a)\xi^2} - e^{-\xi^2}\}\{1 - \cos c\xi\}\, d\xi}{\xi^2(1 - e^{-\xi^2})} \right\}, \qquad 0 < b < a, \tag{22}$$

where

$$Q = F_0 T_1 , \qquad a = T_1/T, \qquad c = R/(\alpha T)^{1/2}. \tag{23}$$

As before, the most interesting quantity is the maximum value of v_P which is attained when $b = a$. Values of this, plotted against T_1/T, are shown in Fig. 3 for various values of the parameter c. For $c < 0.2$ the integrals in (22) do not make an important contribution, while for $c = 1$ they are already close to the limiting case $c \to \infty$ of Fig. 2, Curve II.

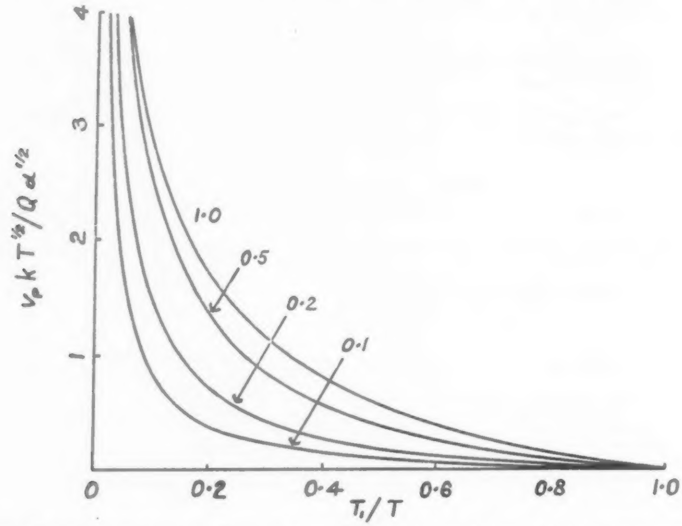

FIG. 3. Values of the periodic temperature at the end of a heating interval for heat supply over a circle of radius R. The numbers on the curves are the values of $R(\alpha T)^{-\frac{1}{2}}$.

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[6]Quart. App. Math., 5, 503 (1947).

4. Pulsed point and line sources in an infinite medium.

For the point source, suppose that heat is supplied at the origin at the rate of q units per unit time in the time intervals

$$nT < t < nT + T_1 , \qquad n = 0, 1, 2, \cdots \tag{24}$$

no heat being supplied at other times. Then, when steady conditions have been attained, the temperature at distance r from the origin at a time bT after the beginning of a heating period is, for $0 < b < a$,

$$\frac{qa}{4\pi kr} + v_P , \tag{25}$$

where the periodic part v_P is

$$v_P = \frac{q}{4\pi kr}\left\{\operatorname{erfc}\frac{C}{2b^{1/2}} - a + \frac{2}{\pi}\int_0^\infty \frac{e^{-b\xi^2}[e^{-(1-a)\xi^2} - e^{-\xi^2}]\sin C\xi}{\xi(1 - e^{-\xi^2})}\,d\xi\right\}, \tag{26}$$

and

$$a = T_1/T, \qquad C = r(\alpha T)^{-1/2}. \tag{27}$$

This is the limiting case of the problem of §3.

For the line source, suppose that heat is supplied along an infinite line at the rate q units per unit length per unit time during the time intervals (24), no heat being supplied at other times. Then, for large values of the time the temperature at distance r from the line is, for $0 < b < a$,

$$-\frac{qa}{4\pi k}\,Ei\left(-\frac{r^2}{4\alpha t}\right) + v_P , \tag{28}$$

where the periodic part v_P is

$$v_P = -\frac{q(1 - a)}{4\pi k}\,Ei\left(-\frac{C^2}{4b}\right) - \frac{q}{2\pi k}\int_0^\infty \frac{e^{-b\xi^2}J_0(C\xi)[(1 - a)e^{-\xi^2} - e^{-(1-a)\xi^2} + a]\,d\xi}{\xi(1 - e^{-\xi^2})}, \tag{29}$$

and a and C are defined in (27), Ei is the exponential integral, and J_0 the Bessel function of the first kind of order zero.

BOOK REVIEWS

Tensor analysis for physicists. By J. A. Schouten. The Clarendon Press, Oxford, 1951. $x + 275$ pp. 50s.

The first five chapters of this extremely well written book are devoted to an exposition of tensor analysis and Riemannian geometry. The author uses his well known expository skill to give a clear geometric interpretation to the various quantities discussed. The last five chapters are devoted to applications of this material to various physical problems. The two groups of chapters are separated by a summary of the salient points of the theory described in the first part of the book. This section should prove to be very useful.

Chapters I and II deal with the algebra of affinors (tensors) and contain a detailed discussion of p-vectors and p-vector densities (anti-symmetric tensors and tensor densities with p indices). The identification of various affinors obtained by restricting the group of the space is made extremely clear.

This discussion is used in Chapter IV to make clear the meaning of Stokes' Theorem which is the name he assigns to the theorem relating integrals over a $q + 1$ dimensional subspace of a space and integrals over the q dimensional boundary of the space.

Chapter V is devoted to a discussion of manifold which have an affine connection and contains in particular a discussion of those parts of Riemannian geometry needed in subsequent applications.

Chapter VI is devoted to a discussion of dimensions of physical objects. The relation between dimensions of physical quantities and the choice of the underlying group is discussed.

The applications of tensor calculus that the author discusses are: Elasticity and peizo-electricity (Chapter VII), Classical dynamics (Chapter VIII), Relativity (Chapter IX). Chapter X on Dirac's matrix calculus gives a brief exposition of the calculus introduced into quantum mechanics by Dirac. Unfortunately the spinor calculus is not discussed.

The space allotted by the author to these applications is relatively small. He has "endeavoured to avoid an incoherent enumeration of interesting facts— and has tried to make each chapter a short but systematic introduction to some branch of theoretical physics." In the main he has succeeded admirably. However, the author devotes more than twice the space allotted to a discussion of graviation to a phenomenological discussion of hydrodynamics. The reviewer feels that the book would have been more useful if this allocation of space were reversed.

In spite of this minor lapse, the book is to be highly recommended for use in a course on tensor analysis and as a reference book.

A. H. Taub

The rise of the new physics. By A. D'Abro. Dover Publications, Inc., New York, 1951. ix + 426 pp. (Volume I), 553 pp. (Volume II). $8.00 the set.

This is a corrected and enlarged edition (two volumes) of The Decline of Mechanism in Physics (1939); it is essentially unchanged except for the addition of 36 portraits of leading physicists and mathematicians. The book is an account of the development of classical physics and approximately the first decade of quantum mechanics. The account has a strong philosophical flavor especially in the first volme. The book is certainly worthwhile for the student of physics and perhaps even more so for the mathematician interested in physics.

Rohn Truell

Mémoire sur l'intégration graphique des équations aux dérivées partielles. Par J. Massau. Edition du Centenaire par les soins du Comité National de Mécanique, Palais des Académies, Bruxelles, Belgique, 1952. x + 544 pp. 200 Belg. frs.

As a matter of general editorial policy, space in these columns is considered too valuable to be available for the review of reprint editions. In the present case, however, there is ample justification for deviating from this policy.

In reprinting this treatise, the Belgian National Committee on Mechanics has rendered a significant service, not only to the memory of an outstanding Belgian scientist, but also to all those who are interested in the application of the method of characteristics to the approximate integration of partial differential equations of hyperbolic type. The volume contains a sequence of papers which originally appeared in the "Annales des Ingenieurs sortis des Ecoles de Gand" around the turn of the century, and which do not seem to be sufficiently well known among scientists working on problems of this kind.

Pages 1 to 58 are devoted to the general theory of characteristics and pages 59 to 120 to the one-dimensional treatment of problems of unsteady flow in canals and rivers. By far the greatest portion of the volume (pages 121 to 391) is concerned with the study of limiting states of equilibrium in a cohesionless soil. A 152-page note on the wave equation concludes the volume.

There is no doubt that acquaintance with this material would have saved much time and effort to the scientists who, during the last twenty years, applied the method of characteristics to numerous problems in hydro-and aerodynamics, detonics, plasticity, soil mechanics, and other branches of mechanics of continua. To mention only a few examples, in the section on soil mechanics the author

establishes the basic geometric properties of the net of failure lines (p. 259), recognizes the possibility of limiting lines (p. 163 ff.), and gives an exhaustive discussion of stress discontinuities (p. 267 ff.). In the theory of plasticity, the corresponding results have been developed *ab initio* whereas they could have been taken almost without modification from Massau's work. The other sections of the book contain similar cases where Massau has anticipated results usually considered to be of much more recent origin.

W. PRAGER

Matter and motion. By J. Clerk Maxwell (Reprinted with notes and appendices by Sir Joseph Larmor). Dover Publications, Inc., New York, 1951. xii + 163 pp. $2.50 clothbound ($1.25 paperbound).

The first edition of this famous little book appeared in 1876 under the imprint of the Society for the Promotion of Christian Knowledge. An American edition, published by D. Van Nostrand Co., was brought out in 1875. The original edition was reprinted with notes and appendices by Sir Joseph Larmor in 1920 and it is this reprinting which has been reproduced in the edition under review.

The book was originally intended by the author to provide in terms of the most elementary mathematics an exposition of the fundamentals of mechanics sufficient to serve as a foundation for the general study of physical science. As such it is still eminently readable for the elementary student. The notes added in the later edition by Larmor help to place the contents in proper perspective with respect to twentieth century revisions of classical physics, notably the theory of relativity.

R. B. LINDSAY

Dialogues concerning two new sciences. By Galileo Galilei. Translated from the Italian and Latin into English by Henry Crew and Alfonso de Salvio. Dover Publications, Inc., New York, 1951. xxi + 300 pp. $1.50 (paperbound).

The publishers are to be congratulated for making this celebrated book once more readily available in the fine translation of Crew and de Salvio, which appeared originally under the imprint of the Macmillan Company in 1914 and was reprinted without change in 1939 by Northwestern University.

The original edition of the Dialogues was published in Leiden in 1638, four years before the author's death, at a time when his difficulties with the ecclesiastical authorities made publication in Italy impossible. It is unnecessary to stress here the importance of the book in the history of physical science. In it the man, acknowledged by all to be the founder of modern physics, sets forth in simple terms the basic concepts of mechanics essentially in the physical form in which they have been used ever since. Every student of elementary physics should read the words of Galileo on the motion of falling bodies and on the mechanical properties of solids and fluids. The present inexpensive edition brings them within the reach of all who can read English.

R. B. LINDSAY

Vektor-und Dyadenrechnung für Physiker und Techniker. By Erwin Lohr. Walter de Gruyter & Co., Berlin W. 35, 1950. xv + 488 pp. DM 24.00.

This is the second edition of a book on vector and dyadic calculus which has apparently enjoyed some popularity in Europe since its publication in 1939. The notation is essentially that of Gibbs, and, as the title indicates, the book is of interest principally to physicists. No extensive alterations of or additions to the first edition are made. The changes consist largely in the addition of a final chapter of miscellaneous applications of vector and dyadic calculus.

The book consists of three parts: I) The arithmetic and algebra of extensive quantities, II) The calculus of extensive quantities, and III) Physical applications. These topics, especially the first two, are covered extensively, but the author's enthusiasm for his subject leads him to apply vector methods to some topics where little is to be gained and dyadics to some where the use of tensors is more satisfactory.

The book will be useful to applied mathematicians and, in particular, to physicists.

WILLIAM PELL

Differential equations. By Robert C. Yates. McGraw-Hill Book Company, Inc., New York, Toronto, London, 1952. vii + 215 pp. $3.75.

This is a text book suitable for an introductory course for engineering students. Equations treated include the various standard types of first order, some special forms of first order, (mostly non-linear), and linear second order with constant coefficients. There is a brief discussion of solutions by numerical methods and by power series, the latter illustrated by the Legendre and Bessel equations.

Partial differential equations are introduced by means of problems in vibrations and heat flow with solutions by separation of variables.

Finally, there is a brief treatment of Fourier series.

The book contains several review chapters inserted at appropriate stages and an abundance of problems with answers.

The illustrative applications of the various equations range over a great variety of physical and geometrical problems, emphasis being placed on first analyzing the problem, then formulating it in terms of a differential equation and finally solving it.

The treatment is non-rigorous throughout and the writing is clear.

George W. Morgan

The scientific papers of James Clerk Maxwell. Edited by W. D. Niven. Dover Publications, Inc., New York. xxix + 801 pp. (two volumes bound as one). $10.00.

This collection of papers is bound as a single volume and contains the papers formerly published in 1890 in two volumes. This edition contains the published papers, lectures, and addresses of Maxwell and probably includes all of his work outside of his texts on "Theory of Heat", "Electricity and Magnetism", and "Matter and Motion".

While it may be superfluous to list the contents or to review something published in 1890, it is, I believe, not out of place to recommend as very interesting reading almost all of these papers by the first Cavendish Professor of Experimental Physics.

Rohn Truell

Proceedings of the Second Canadian Mathematical Congress, Vancouver, 1949. University of Toronto Press, Toronto, Canada, 1951. xxxi + 255 pp. $6.00.

This volume contains the papers presented at the Second Canadian Mathematical Congress, held at Vancouver, B. C., from September 5–9, 1949. The eighteen papers presented cover a wide variety of topics. Historical and survey papers are presented as well as technical work on pure and applied mathematics.

The papers included are: A. Zygmund, *Polish Mathematics between the two wars;* P. A. M. Dirac, *The relation of Classical to Quantum Mechanics;* A. W. Conway, *Hamilton, his life, work, and influence;* H. J. Bhabha, *Recent scientific developments in India;* G. Szegö, *Principal frequency, torsional rigidity and electrostatic capacity;* L. Schwartz, *Les Mathématiques en France pendant et après la guerre;* W. H. Watson, *The role of the National Research Council in the education of Canadian mathematicians and physicists;* B. Jessen, *Mean motions and almost periodic functions;* R. L. Jeffery, *Non-absolutely convergent integrals;* G. F. D. Duff, *A development in the theory of the F-Equation;* Josephine Mitchell, *An example of a complete orthonormal system and the kernel function in the geometry of matrices;* P. Mandl and J. R. Pounder, *Wind tunnel interference on rolling moment of a rotating wing;* R. De Vogelaere, *Une nouvelle famille d'orbites périodiques dans le problem de Störmer: les ovales;* B. Davison, S. A. Kushneriuk, and W. P. Seidel, *Influence of a small black cylinder upon the neutron density in an infinite non-capturing medium;* W. P. Thompson, *Thermal convection in a magnetic field;* E. Leimanis, *The application of infinitesimal transformations to the integration of differential equations of exterior ballistics by quadratures;* A. E. Scheidegger, *On gravitational radiation;* M. A. Melvin, *Symmetry and affinity of electromagnetic fields, charges, and poles.*

W. H. Pell

Advanced engineering mathematics. By C. R. Wylie, Jr. McGraw-Hill Book Co., Inc., New York, Toronto, London, 1951. xiii + 640 pp. $7.50.

This text book for third or fourth year engineering students treats material of proven value to the analytical engineer. The main mathematical topics are: linear differential equations with constant coefficients, Fourier series, the Laplace transformation, separable partial differential equations, Bessel functions, complex variables, conformal mapping, vector analysis, and numerical analysis. In the development of these topics the relation to physical problems is constantly brought out. Thus, analytic functions are related to fluid mechanics. A valuable feature is a chapter on electrical and mechanical vibrations which explains electrical circuits and how to set up electromechanical analogies. There are an extremely large number of diagrams and worked out examples; this should serve to make the book easy to read.

R. J. Duffin

Tensor analysis. By I. S. Sokolnikoff. John Wiley & Sons, Inc., New York and Chapman & Hall, Ltd., London, 1951. ix + 333 pp. $6.00.

This text should furnish a good introduction to tensor analysis for students on the senior-graduate level. The exposition is clear and is illustrated by examples. The subject matter is arranged so that the formal processes of tensor analysis are developed and then applications are made to geometry, analytical dynamics, relativity, and the mechanics of continuous media. In particular, the author's development of the theory of non-linear elasticity should prove of interest. There are a few minor errors and misprints.

The text covers a considerable amount of material which may be subdivided into: (1) matrix theory; (2) the theory of tensor analysis; (3) applications of tensor analysis. In the first part, the author discusses: linear vector spaces, linear transformations in terms of matrices, characteristic values and the reduction of two quadratic forms. The tensor theory is developed by introducing the group of coordinate transformations and studying the induced group of transformations for the covariant and contravariant components of tensors. This is followed by a study of tensor algebra, the metric tensor, the Christoffel symbols, covariant differentiation of tensors, the e-systems and generalized Kronecker deltas, and the Riemann-Christoffel tensor. In the applications of tensor analysis to geometry, the above results are interpreted in terms of curvilinear coordinate systems and base vectors in Euclidean three space. Parallelism of vectors, in three space, the Frenet formulas for curves, and an introduction to surface theory (imbedded in three space and intrinsic) are discussed. In the sections on analytical dynamics and relativity, the author considers the Lagrange equations of motion of a system of particles, the principle of least action, Hamilton's equations, some elements of potential theory, the Lorentz-Einstein transformation law of restricted relativity, and the Schwarzchild line element. In the final chapter the mechanics of continuous media are studied by following the general approach of F. D. Murnaghan, Finite Deformations of an Elastic Solid, Am. Journal of Math. vol. 59, 1937. This leads to a unified and interesting development of the strain tensor in non-linear elasticity.

N. Coburn

Conformal mapping. By Zeev Nehari. First Edition. McGraw-Hill Book Company, Inc., New York, Toronto, London, 1952. viii + 396 pp. $7.50.

Professor Nehari's clearly written and thorough monograph on conformal mapping can be read with profit by all mathematicians who have an interest in this important subject. His very detailed study of special conformal transformations, and their analytical expressions, fills a real gap in the literature. Especially to be commended are the elegant discussions of circular polygons and hypergeometric functions, distortion theorems, the symmetry principle, and of elliptic functions and the Picard theorem. The introductory discussion of multiply connected domains is also useful.

The style of the book is modern, and all discussions very rigorous and complete. Because of this

fact, and the presence of ample exercises, it should serve as an excellent text for a course on conformal mapping. Because of its specialized emphasis, it would seem more useful as a reference than as a basic text for a course on complex variable theory.

Physicists and engineers who are interested in transformations of special domains of the classical type, should also find the book an excellent reference. (Other references for this purpose are A. Betz, "Konforme Abbildung" and H. Kober's "Dictionary of Conformal Transformations," published by the British Admiralty Computing Service.) Those who desire a sound basic understanding of the theory of conformal transformations, can hardly do better than study it carefully.

On the other hand, there is little discussion of numerical methods applicable to general domains; reductions to tabulated real functions and problems of determining numerical parameters are not considered; and the applications to non-Euclidean geometry and to problems of electromagnetism and fluid mechanics are hardly mentioned. For these reasons, and because of its general tone (viz., the proof on p. 278 of the elementary formula for tan $(z + \zeta)$, and the discussion of $z = \frac{1}{2}(\zeta + \zeta^{-1})$ as a special case of $z = (a\zeta^2 + b\zeta + c)/(d\zeta^2 + e\zeta + f)$), the book is less suitable as an "applied" mathematics text.

However, "applied" mathematicians already familiar with practical aspects of the subject, and prospective mathematical analysts, will find the volume a nearly ideal reference, both for the general theory of conformal transformations, and for the treatment of special cases.

Garrett Birkhoff

Foundations of high-speed aerodynamics. Facsimiles of nineteen fundamental studies as they were originally reported in the scientific journals. With a bibliography compiled by George F. Carrier, Professor of Engineering, Brown University. Dover Publications, Inc., New York, 1951. 286 pp. $3.50.

In addition to an extended bibliography, the book contains photo-offset reproductions of nineteen basic papers on the dynamics of compressible fluids. Obviously, space limitations have excluded some important contributions to the subject; otherwise, the collection illustrates well the rapid development of this field. The bibliography is arranged according to the following headings: Hodograph Method. The Rayleigh-Janzen Method. The Prandtl-Glauert Method. Supersonic Flow. Shock Waves. Boundary Layer. The Oscillating Airfoil and Other Unsteady Flow Phenomena. General.

W. Prager

The theory of electromagnetic waves. (A Symposium). Interscience Publishers, Inc., New York, 1951. vii + 389 pp. $6.50.

This book is a series of papers presented at a symposium at Washington Square College of New York University in June 1950. The symposium was sponsored by the Air Force Cambridge Research Laboratories and New York University. This book contains eighteen complete papers and three abstracts of papers as follows: On the Theory of Electromagnetic Wave Diffraction by an Aperture in an Infinite Plane Conducting Screen (37pp.), by H. Levine and J. Schwinger. On Systems of Linear Equations in the Theory of Guided Waves (18 pp.), by W. Magnus and F. Oberhettinger. Wiener-Hopf Techniques and Mixed Boundary Value Problems (16 pp.), by S. N. Karp. Asymptotic Solutions of a Differential Equation in the Theory of Microwave Propagation (12 pp.), by R. E. Langer. Criteria for Discrete Spectra (11 pp.), by K. O. Friedrichs. Extension of Weyl's Intregral for Harmonic Spherical Waves to Arbitrary Wave Shapes (10 pp.), by H. Poritsky. Kirchhoff's Formula, Its Vector Analogue, and Other Field Equivalence Theorems (17 pp.), by S. A. Schelkunoff. On the Diffraction Theory of Gaussian Optics (14 pp.), by H. Bremmer. Diffraction and Reflection of Pulses by Wedges and Corners (20 pp.), by J. B. Keller and A. Blank. Vector Wave Functions (10 pp.), by R. D. Spence and C. P. Wells. The W. K. B. Approximation as the First Term of a Geometric-Optical Series (21 pp.), by H. Bremmer. Remarks Concerning Wave Propagation in Stratified Media (12 pp.), by S. A. Schelkunoff. The Theory of Magneto Ionic Triple Splitting (32 pp.), by O. E. H. Rydbeck. An Asymptotic Solution of Maxwell's Equations (38 pp.), by Morris Kline. Field Representations in Spherically Stratified Regions (53 pp.), by N. Marcuvitz. Propagation in a Non-homogeneous Atmosphere (34 pp.), by B.

Friedman. Reflection of Electromagnetic Waves from Slightly Rough Surfaces (28 pp.), by S. O. Rice. The Theory of Scattering of Radio Waves in the Troposphere and Ionosphere (abstract), by H. G. Booker. Properties of Guided Waves on Inhomogeneous Cylindrical Structures (abstract), by R. B. Adler. Evaluation of Integrals Associated with Wave Motion in Dispersive Media and the Formation of Transients (abstract), by M. Cerrillo. Electromagnetic Research in the U. S. Air Force Research Program, by N. C. Gerson.

This book is obviously a discussion of mathematical methods applied to special problems in propagation, diffraction, dispersion, etc., of electromagnetic waves. The book will undoubtedly be of value to the student of advanced electromagnetic theory and to the research worker.

ROHN TRUELL

Advanced calculus. By Wilfred Kaplan. Addison-Wesley Press, Inc., Cambridge, Mass., 1952. xiii + 679 pp. $8.50.

This text on advanced calculus is an excellent addition to the list of books on this subject, because of its laudable emphasis on vector methods in all phases of the calculus in which they can be applied.

The material contained covers all the usual subjects found in a course of advanced calculus, but the emphasis seems definitely slanted toward the advanced engineer or the applied mathematician. The topics discussed include differential and integral calculus of functions of several variables, infinite series, Fourier series and orthogonal functions, functions of a complex variable, and ordinary and partial differential equations. In addition, there is an excellent treatment of vectors and their properties, vector differential calculus, and vector integral calculus, both in the plane and in space. The consistent use of vectors is well illustrated in the discussion of orthogonal functions, where the functions in question are thought of as elements in a vector space. Thus, the engineering student, to whom the concept of a general set of orthogonal functions and their properties may appear as an intellectual abstraction, will be able to relate these functions to what is to him a more or less familiar concept, vectors.

There is a large number of problems of which a goodly portion is of the so-called applied type, i.e., relating to the fields of mathematical physics, hydromechanics, elasticity, potential theory, etc. The proofs of some of the theorems are given as problems. The rest of the problems, though not trivial, are of the standard drill type. Answers are given for all problems which are not self-contained.

Some of the longer and more difficult results are given without proof, but whenever this is done, references are given where the proofs may be found. As the author states, "A competent teacher can easily fill in these gaps, if so desired, and thereby present a complete course in real analysis." Beyond this, the work is completely rigorous, clearly presented, and concisely written.

At the end of each chapter there is a long list of suggested supplementary reading.

HARRY J. WEISS

The principle of relativity. By H. A. Lorentz, A. Einstein, H. Minkowski and H. Weyl. Notes by A. Sommerfeld. Translated by W. Perrett and G. B. Jeffrey. Dover Publications, Inc., New York, 1951. viii + 216 pp. $3.50 (clothbound) and $1.50 (paperbound).

This collection of the classical papers by the above-named authors has been made available to the American market at a very attractive price and with the added convenience of being in English throughout. This edition will make it possible for graduate students so inclined to read the key papers that established relativity, and about as easily as they could get the same material out of a text book on the subject. One can only regret that this collection has not been supplemented by the most important papers that have appeared since, but such additions would undoubtedly have driven the price up to the point where purchase by students would no longer be feasible.

It is interesting to note that the so-called Lorentz transformation is not due to Lorentz at all (I owe this remark to Dr. H. Zatzkis of the University of Connecticut) cf. page 14. Lorentz correctly describes the deformation of moving scales, but the transformation law for the time is quite different from the one postulated by Einstein (page 48).

Einstein's fundamental paper on the general theory of relativity occupies some fifty pages in this

small format. In a modern U.S. physics journal it would probably take about twenty pages. In that amount of space, Einstein manages to give a comprehensive introduction to the general theory of relativity to the extent he had then been able to develop the theory, including an almost complete presentation of the foundations of Riemannian geometry. Very commonly the authors of today's theoretical papers excuse their lack of comprehensibility on the grounds that the editor will not let them use enough space. This paper by Einstein, covering the complete foundations of a new theory, presenting all of its physical and mathematical aspects, is "experimental" proof that within the length of a paper perfectly acceptable to today's editors it is possible to write lucidly, provided the author is willing to go to the trouble of organizing his material with the reader in mind. It has been this reviewer's privilege to see Professor Einstein at work on several papers intended for publication and to watch the infinite pains with which he went over the drafts, sentence by sentence, until they were satisfactory to him. The results are well worth the effort.

The volume concludes with a paper by H. Weyl, which now has mostly historical significance. In this paper Weyl presented his generalization of Riemannian geometry, in which the length of a displacement vector is not an invariant quantity. Occasionally, one reads in the literature references to Weyl's geometry which convey the impression that Weyl's geometry reduces the *length* of a vector to a non-integrable status in the same way that Riemannian geometry makes the direction of a parallel displaced vector non-integrable. This impression is not quite correct, insofar as Weyl's geometry permits the construction of vectors (those of weight-½) whose norm is invariant. It is true, though, that the elementary displacement vector dx has no invariant norm. Unfortunately, in spite of its intrinsic conceptual beauty, Weyl's geometry has not led to a viable physical theory. This may have been the fault of those trying to apply it, including Weyl himself, and it is conceivable that another attempt may succeed.

Altogether, this collection of original articles may be warmly recommended to those who find the German original articles a bit too difficult and who, nevertheless, appreciate the value of studying the source material. Undoubtedly, many readers will find it worthwhile to expand their reading beyond the papers available here. There are numerous references (such as to the original papers by Michelson and Morley, which were, of course, published in English), but no index.

PETER G. BERGMANN

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